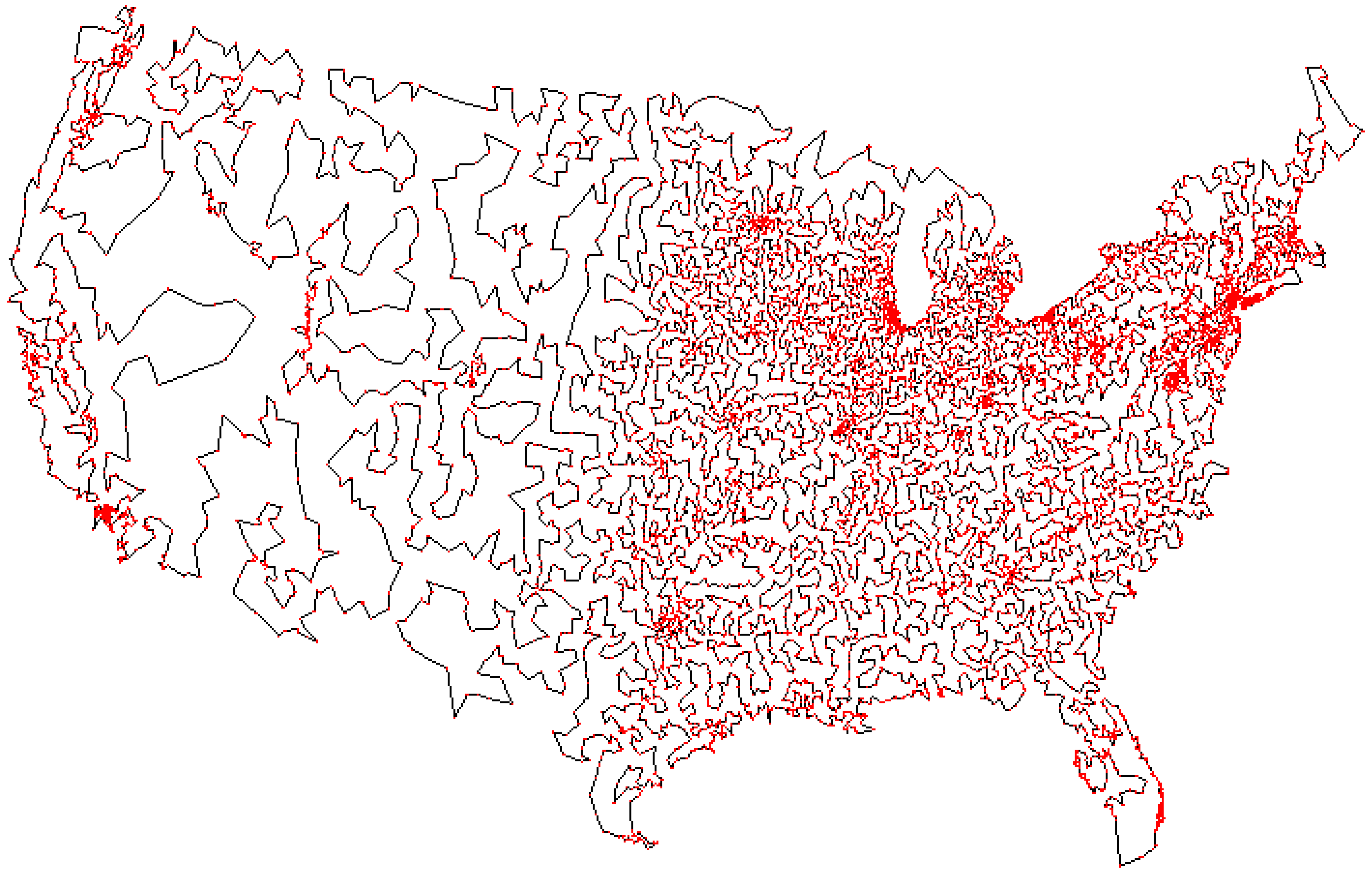
PTAS for ETSP

Polynomial-time approximation scheme for the Euclidean Traveling Salesperson Problem

The Problem

Euclidean Traveling Salesperson Problem (ETSP)

Input: set of points P in $\mathbb{R}^2$



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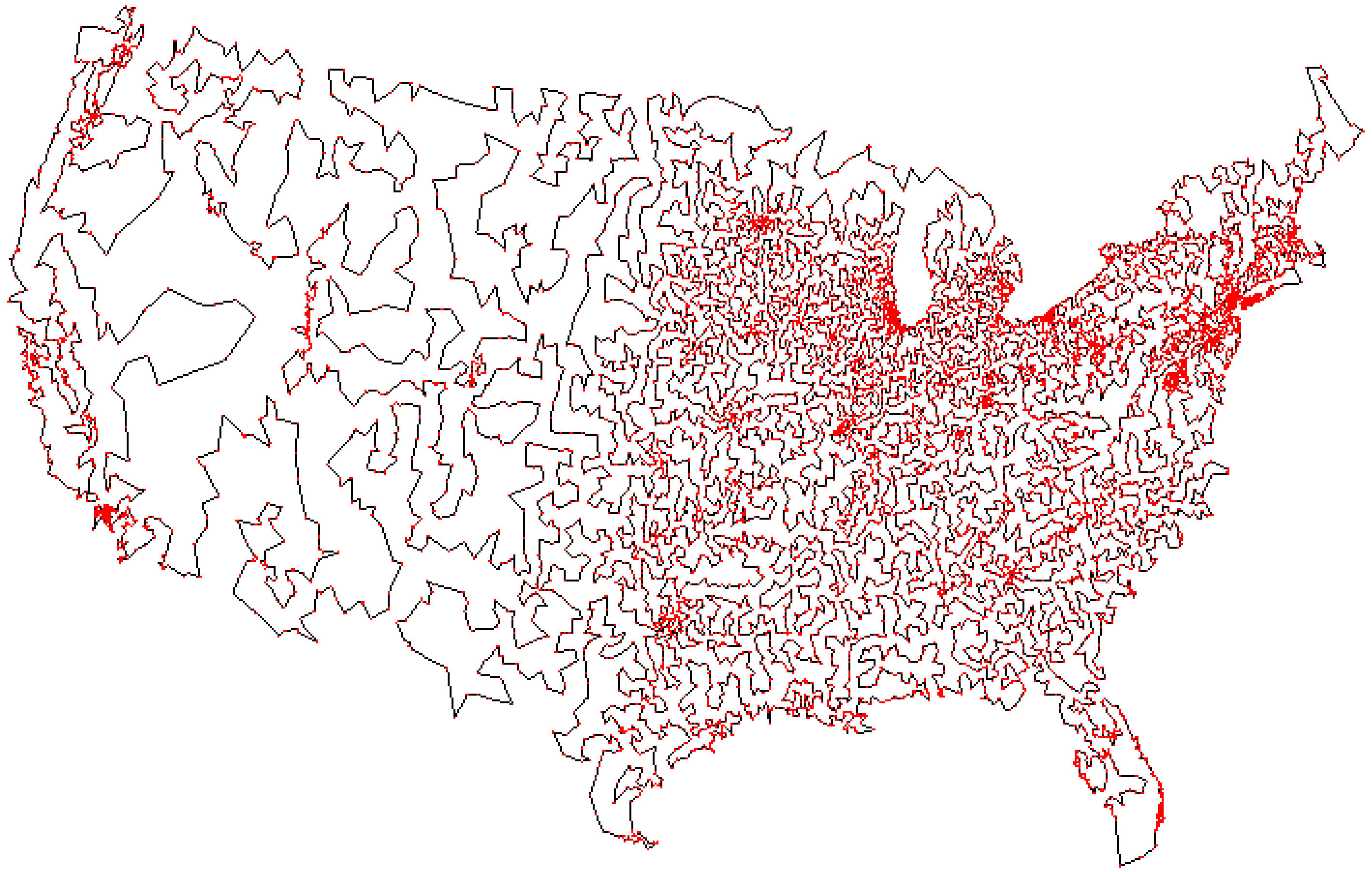
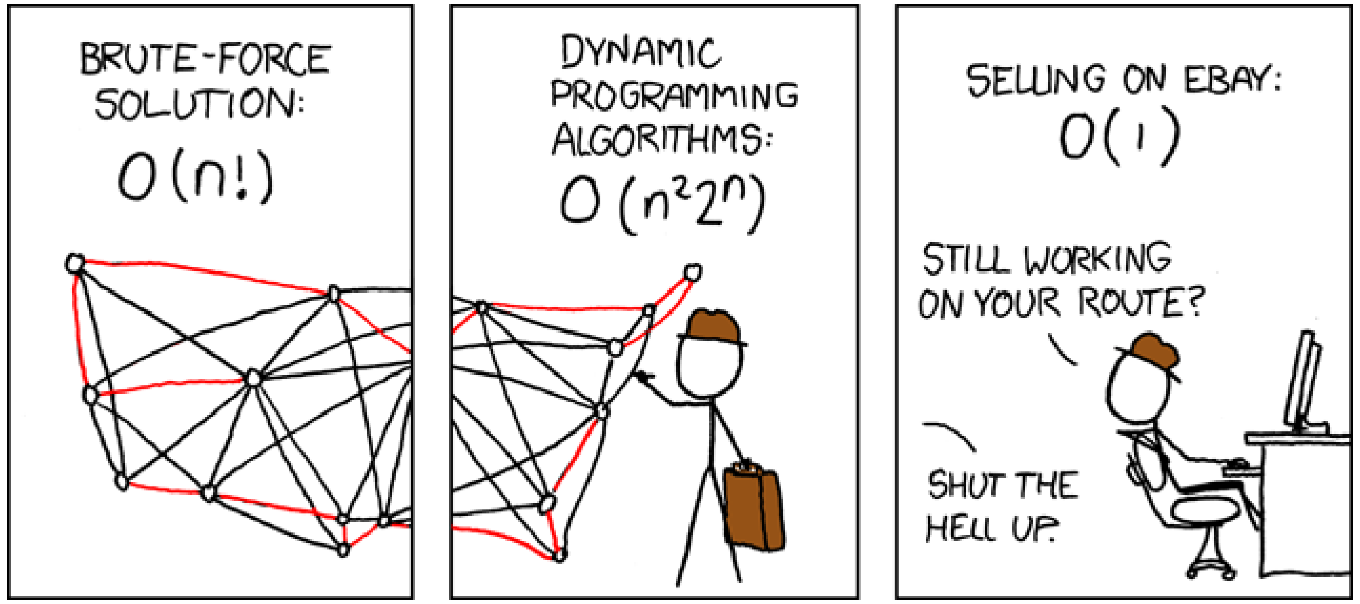


Today: A polynomial-time approximation scheme (PTAS) for the ETSP

Gives for any fixed $\varepsilon > 0$ a polynomial-time $(1+\varepsilon)$-algorithm

Motivation

TSP: classic NP-hard optimization problem



https://xkcd.com/399/

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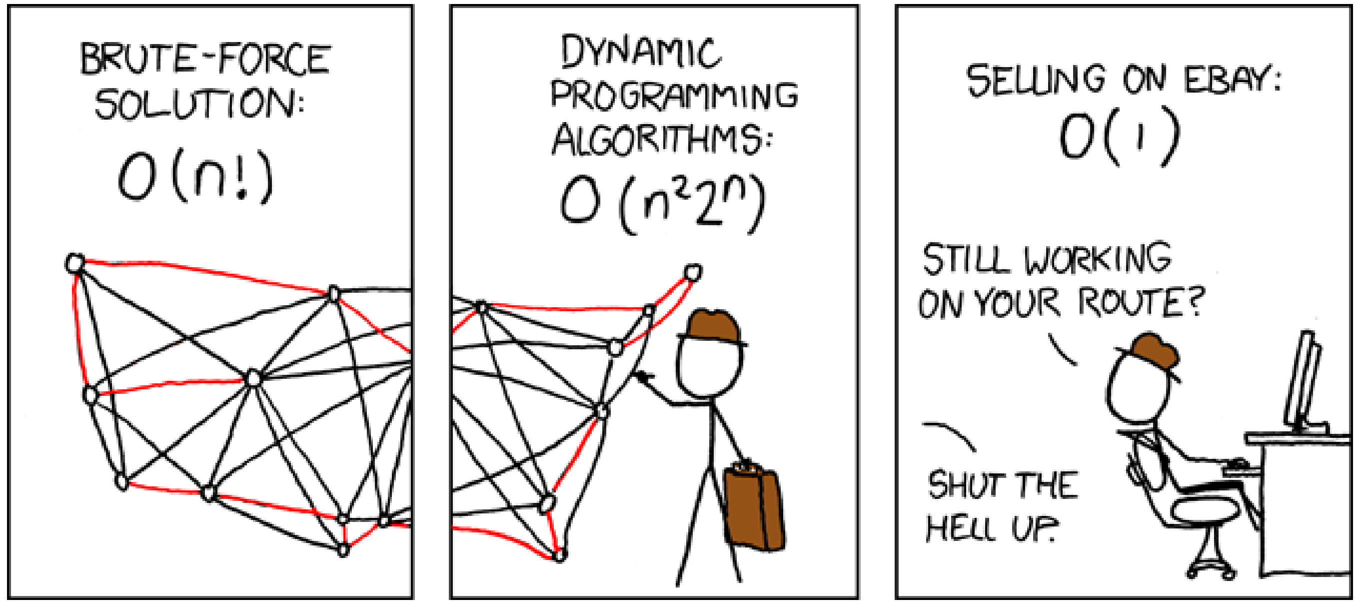

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Polynomial time approximation algorithms for TSP

- for general TSP: not possible

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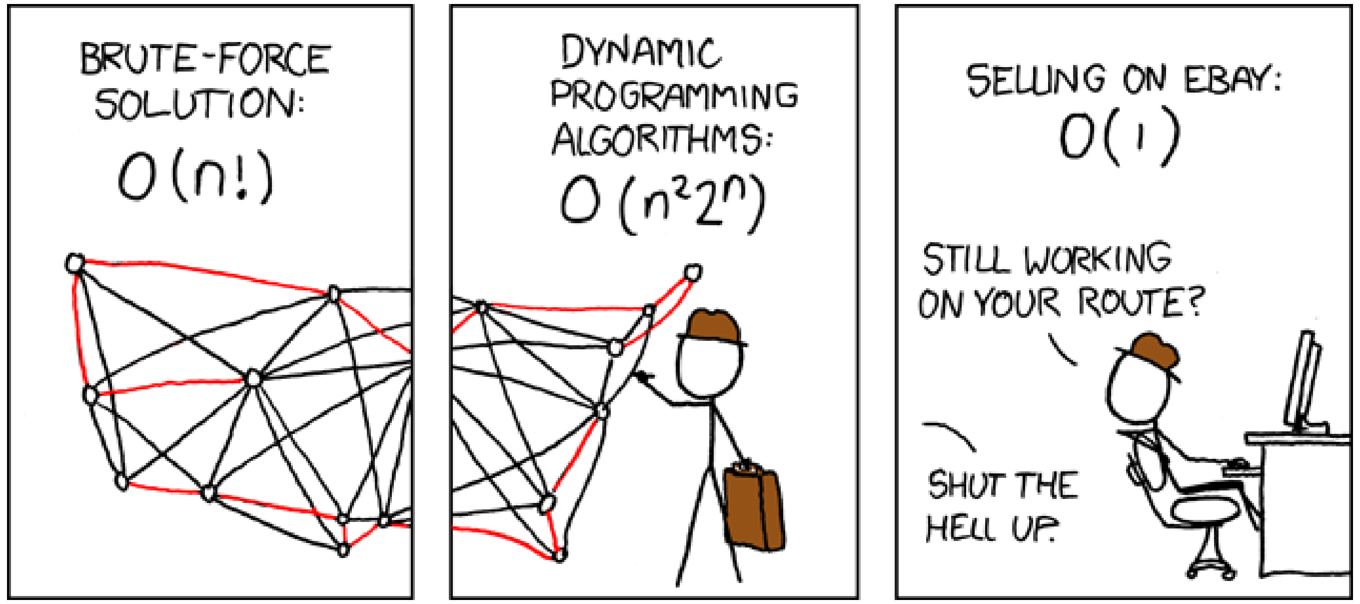

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- for general TSP: not possible
- for metric TSP: not possible with approximation factor $< 123/122 \approx 1.008$, Christofides: 1.5-approximation, first improved in 2020: $1.5 - 10^{-36}$

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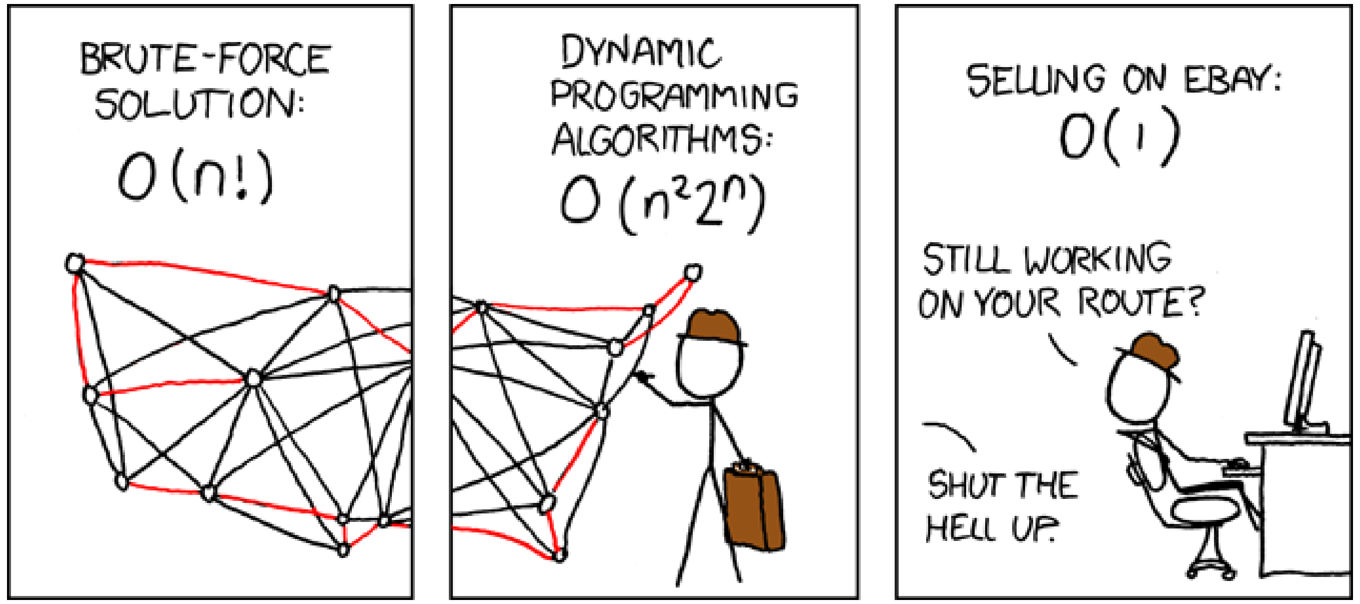

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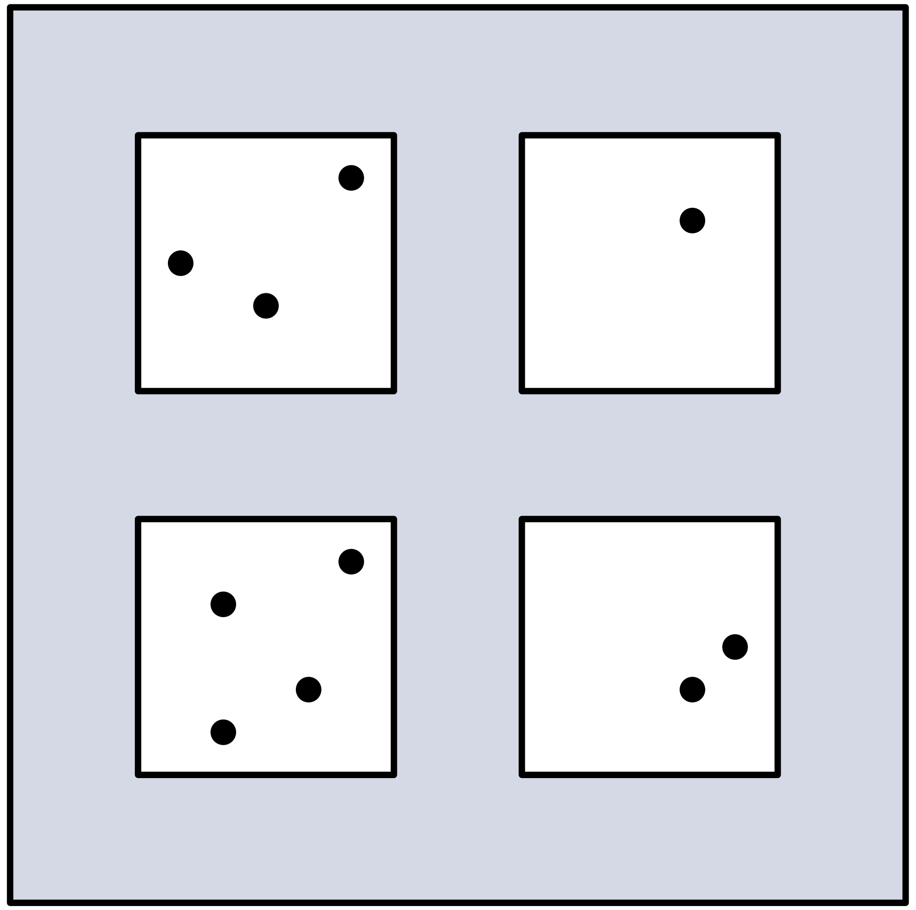
- for ETSP: PTASs by Sanjeev Arora [1998] and Joe Mitchell [1999] $\rightarrow$ Gödel prize in 2010
  today: Arora's algorithm

Overview

1. Intuition
2. Subproblems
3. Algorithm
4. Running time
5. Quality of approximation

Intuition - Approach

Dynamic programming on quadtrees

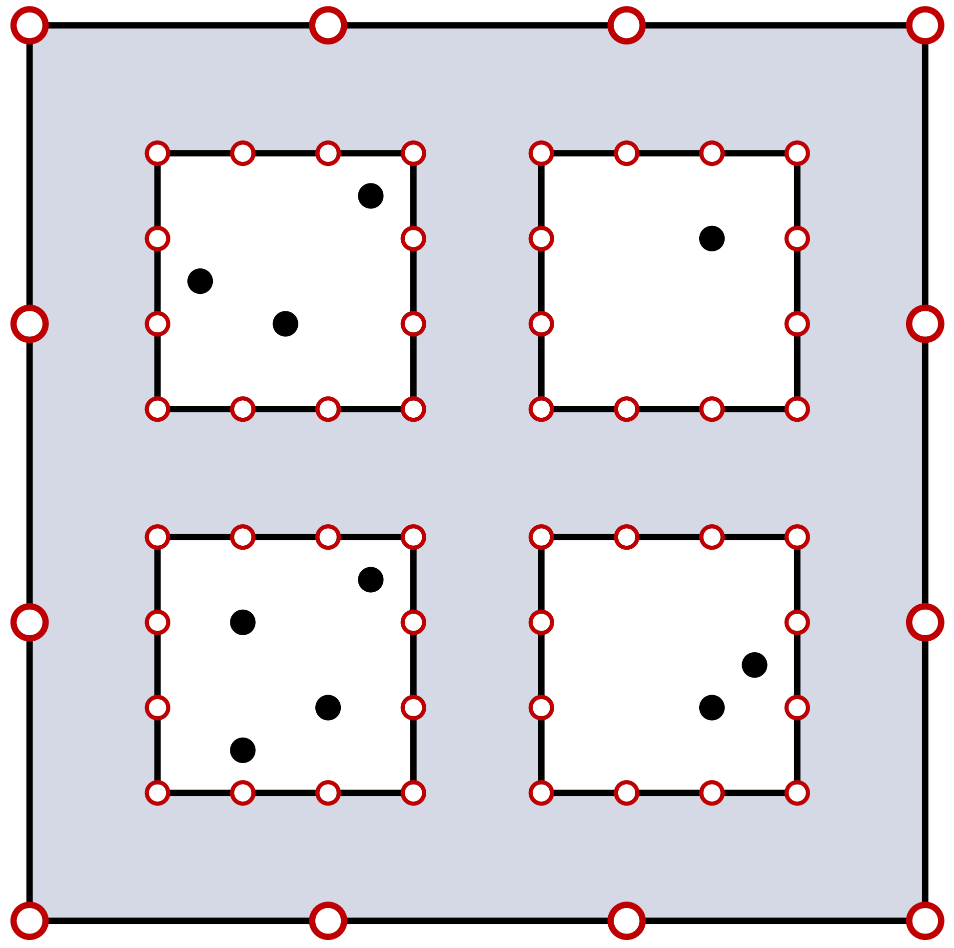


Intuition - Approach

Dynamic programming on quadtrees

Portals on the boundaries

Evenly placed and on each corner



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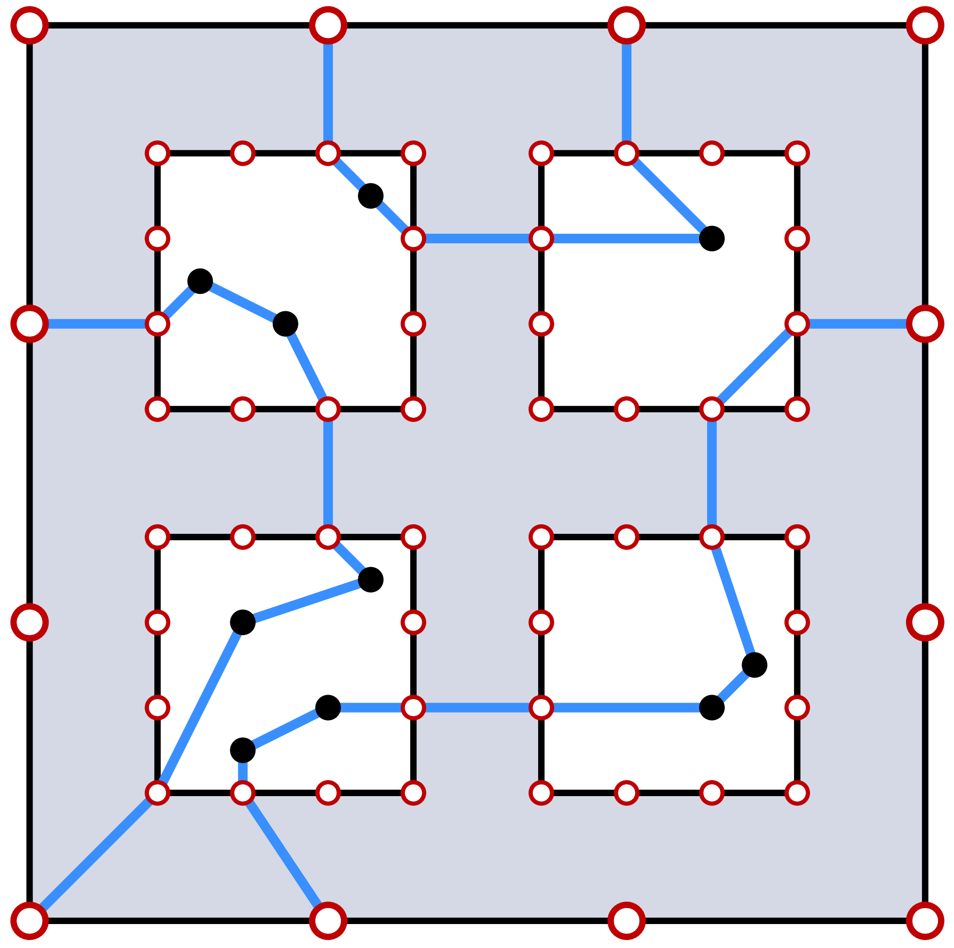

Intuition - Approach

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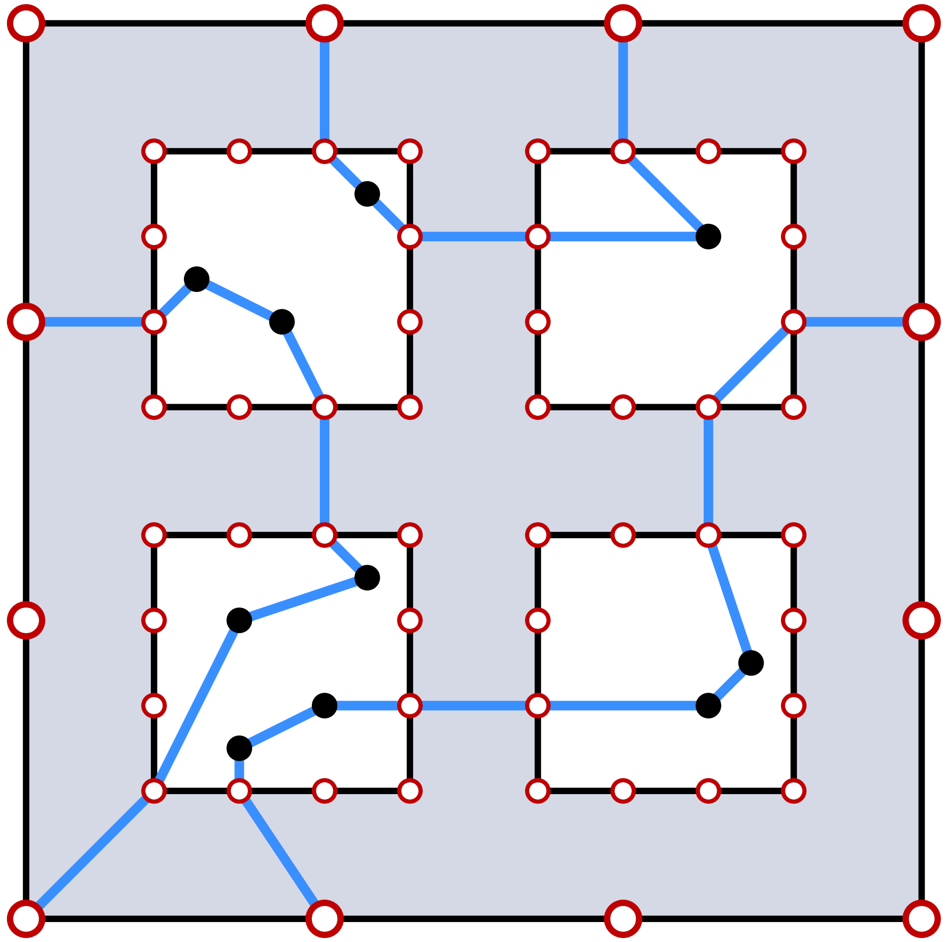
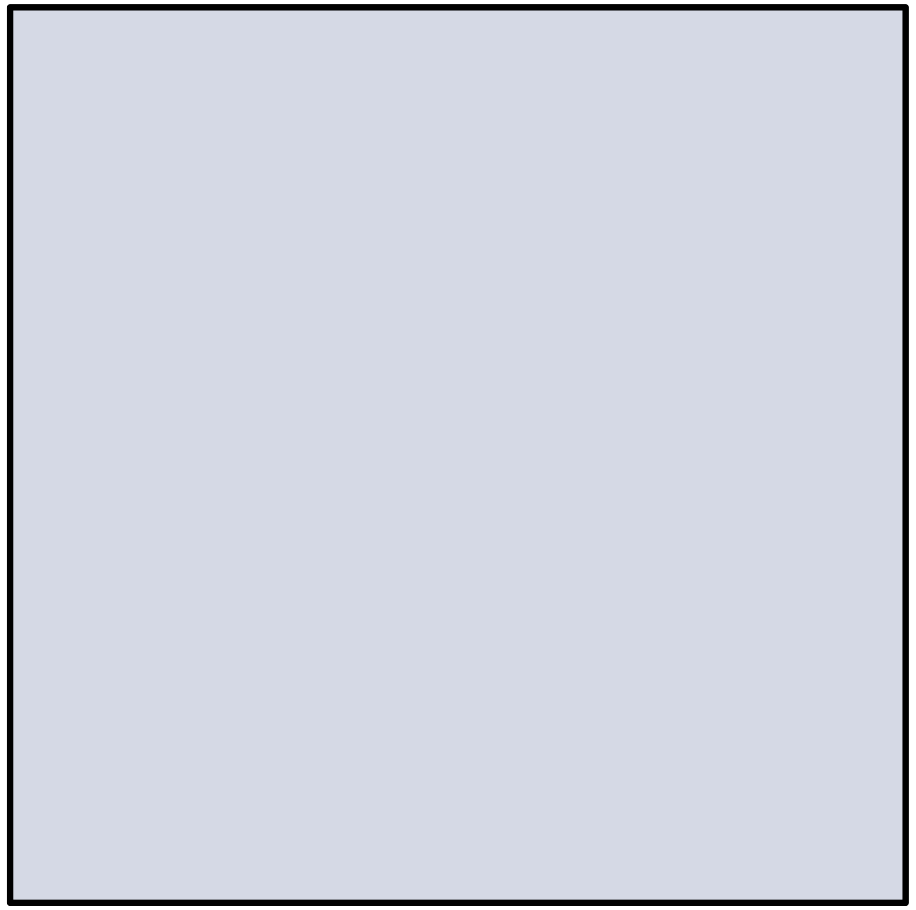
What defines a subproblem?



Subproblems

Square S



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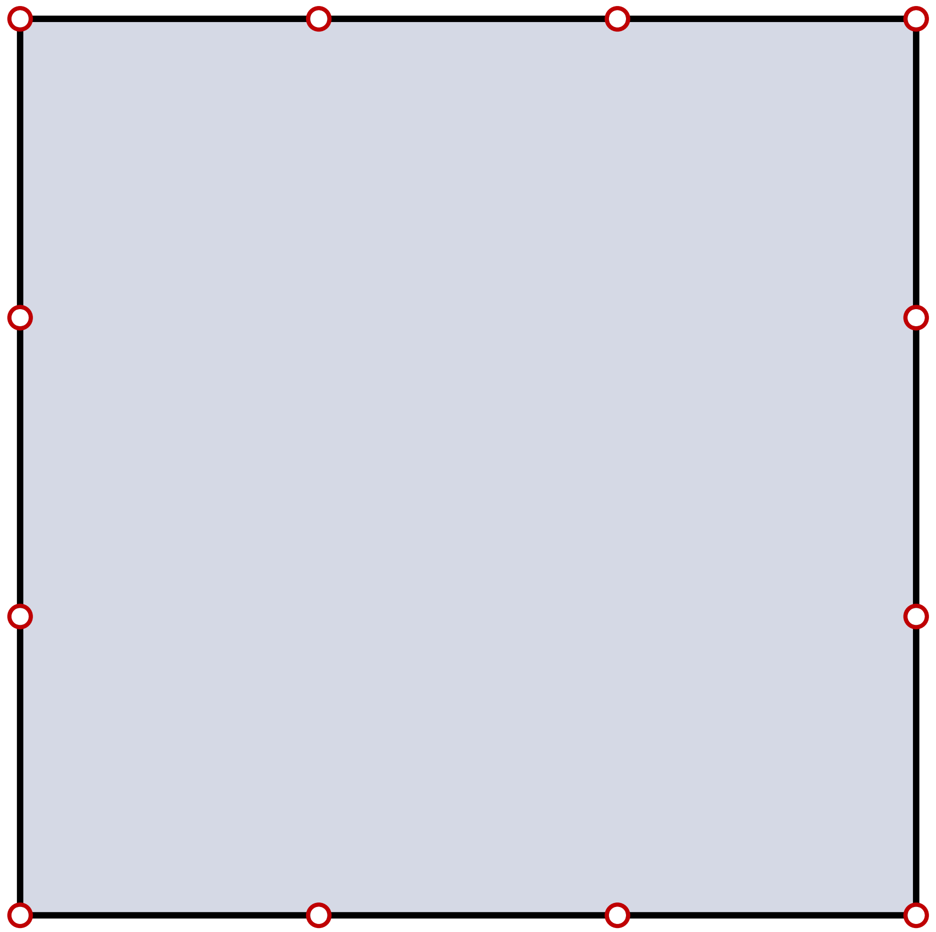
Square S

The $m + 2$ portals on each edge of S



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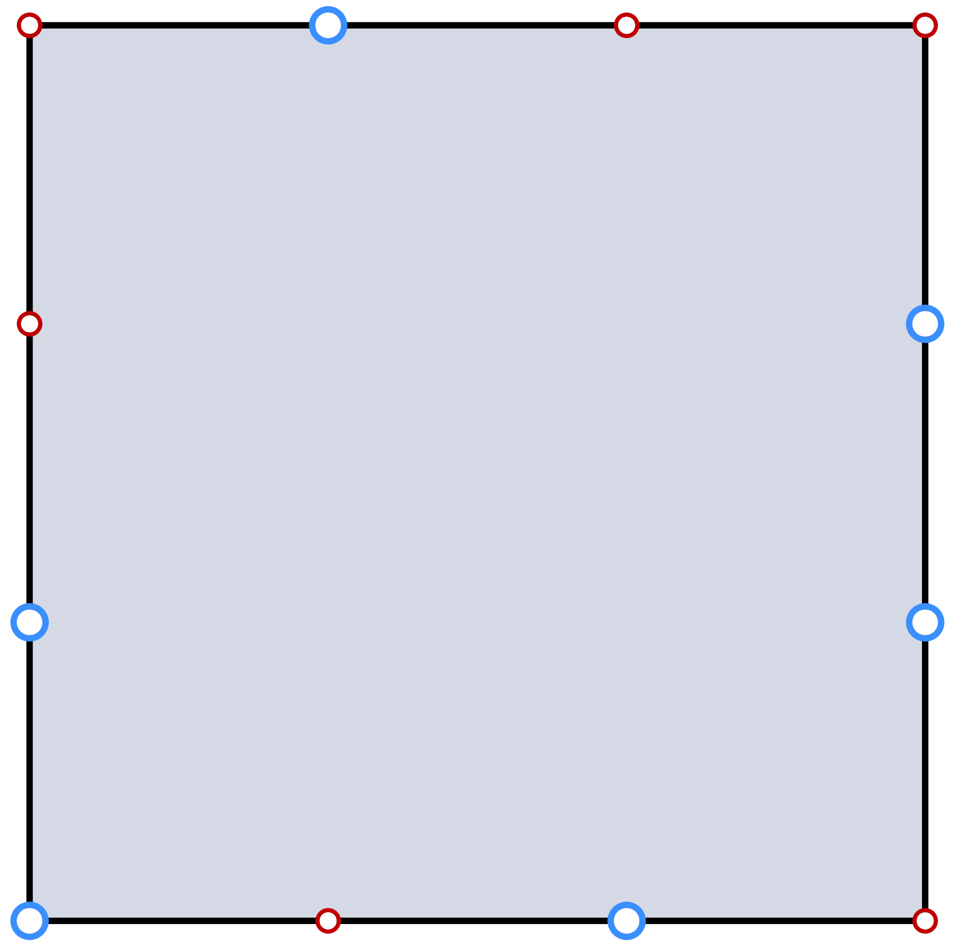

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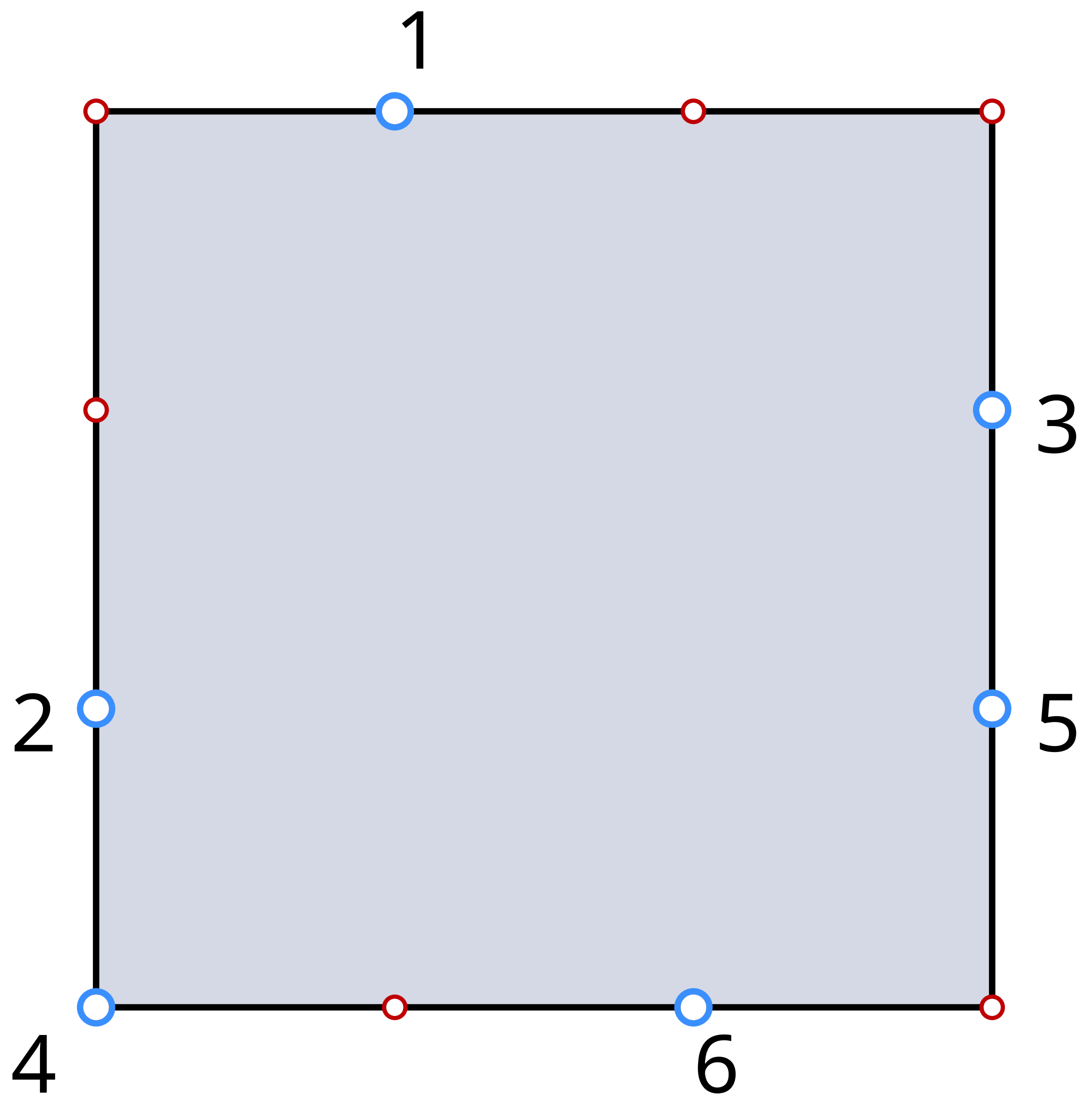

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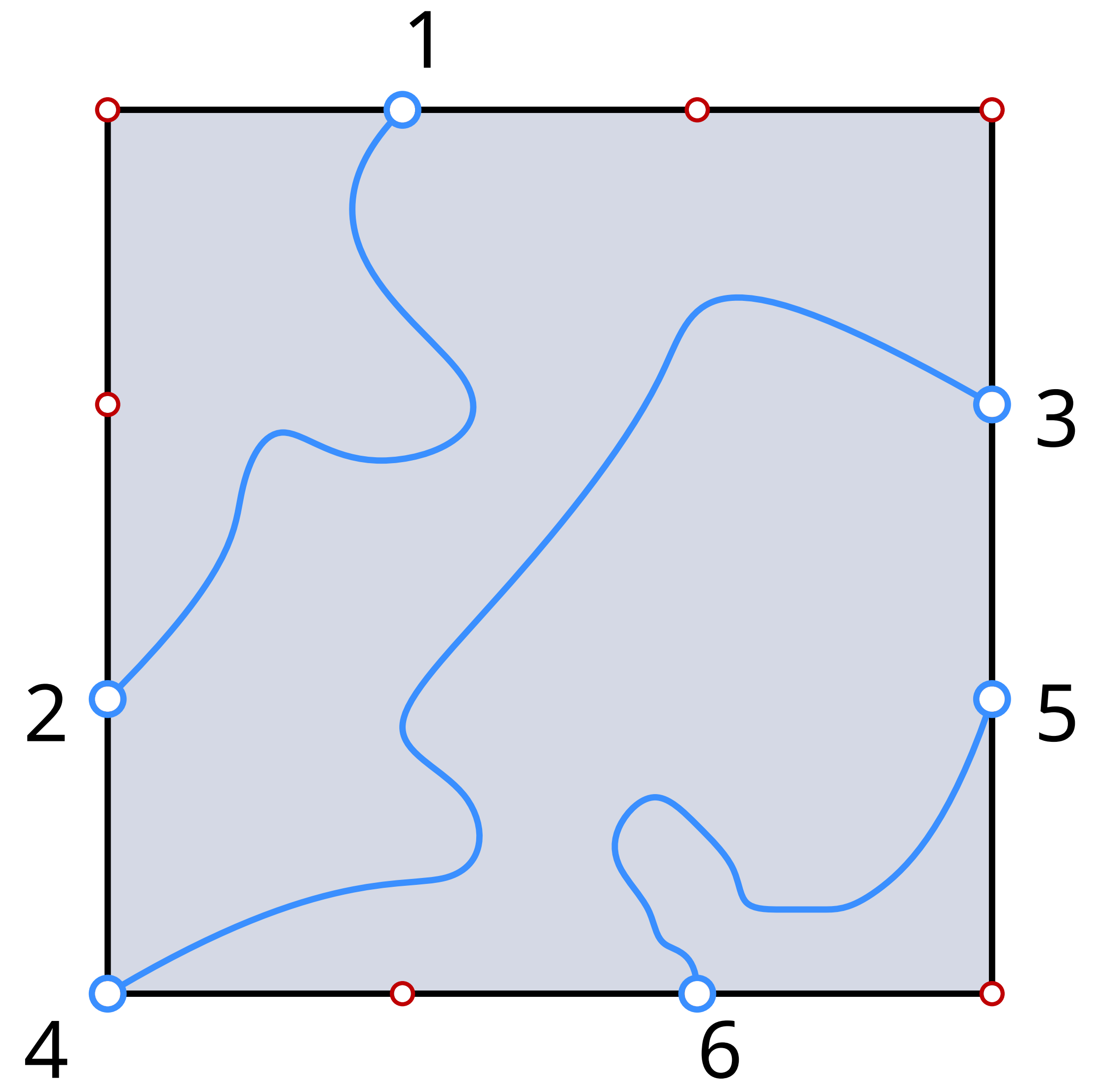
Subproblem denoted (S, M)

Solution: length of the shortest partial tour abiding by M



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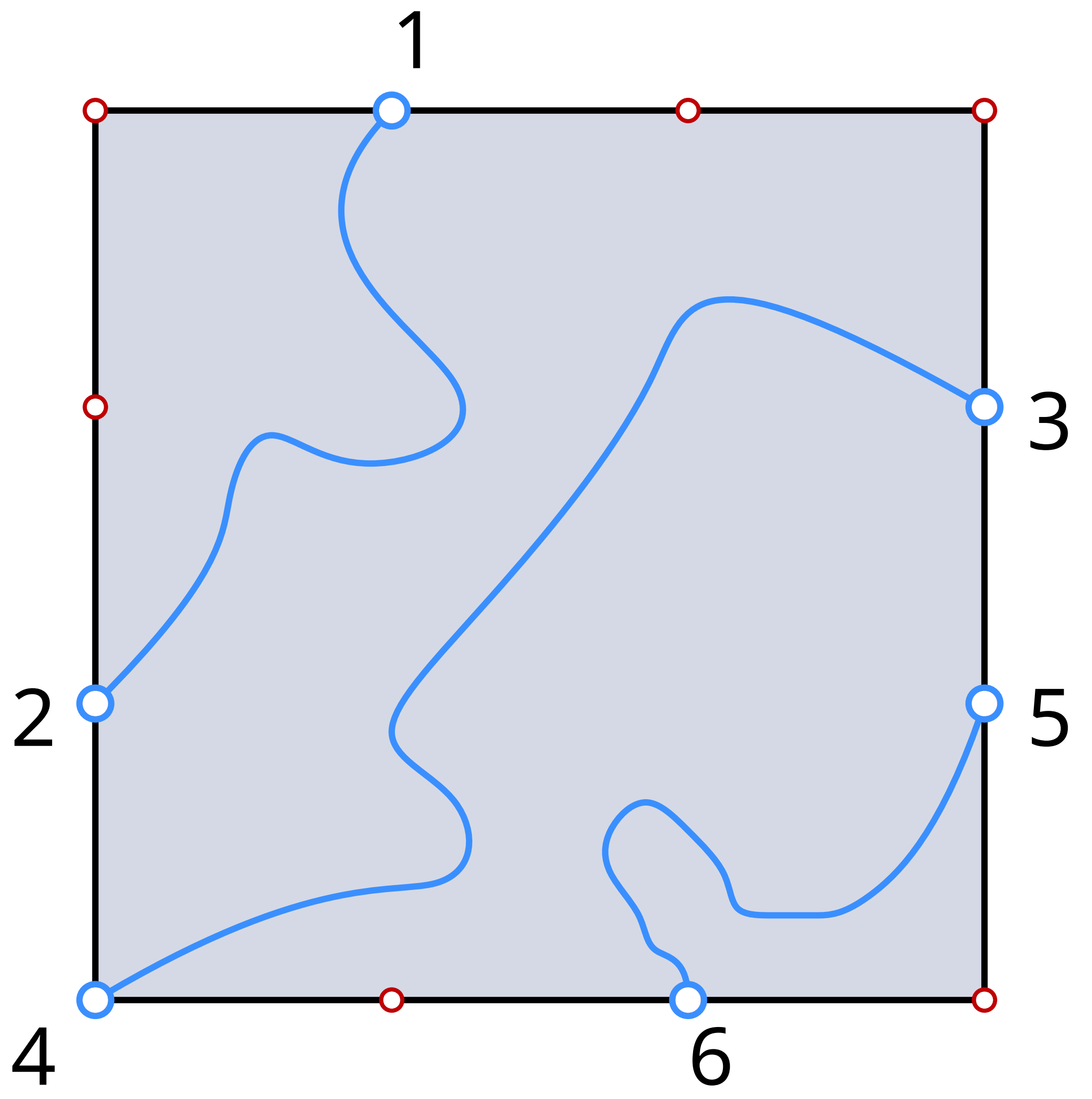

Restrictions:

The solution takes each portal at most twice (patching lemma $\to$ later)

Per side of S at most k portals can be used

Overview

1. Intuition ✓
2. Subproblems ✓
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Problems with constructing a quadtree on P directly?

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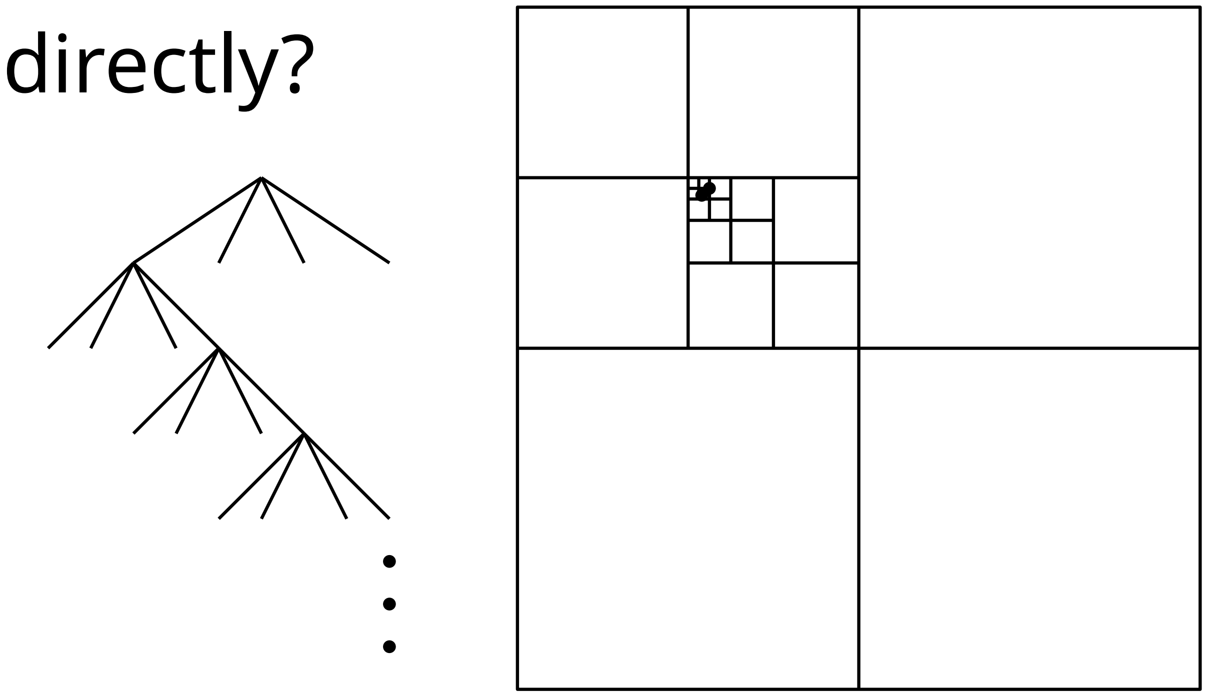

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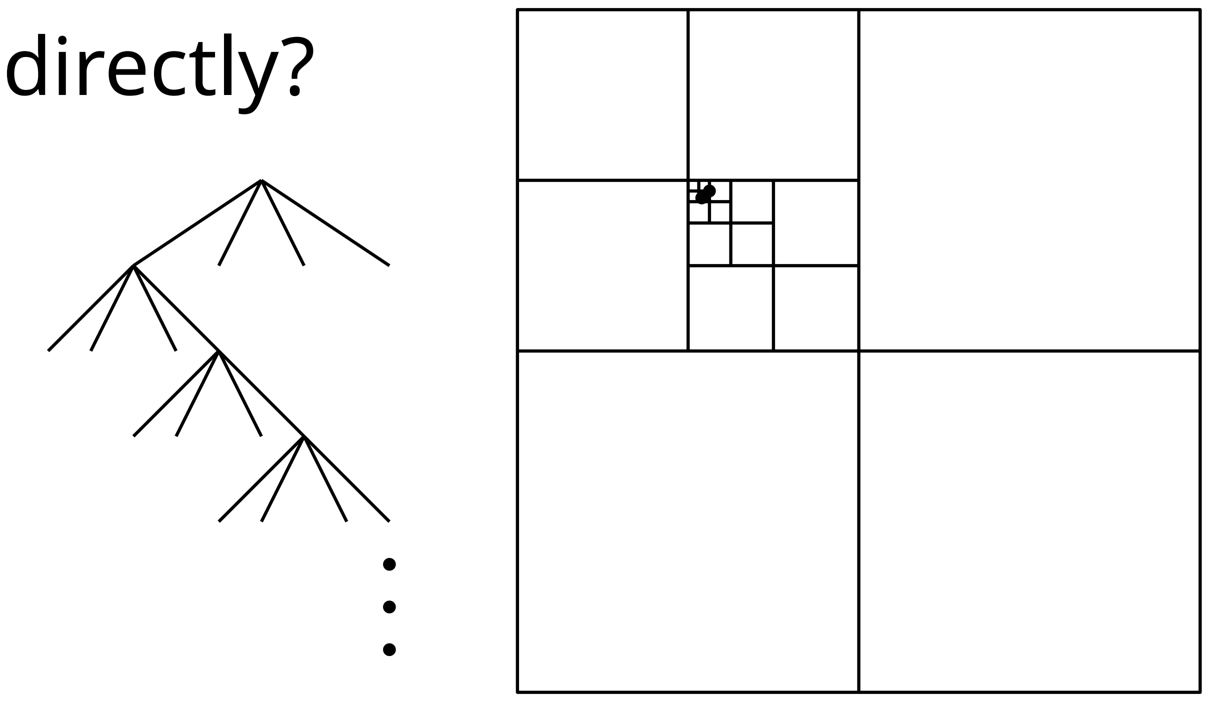
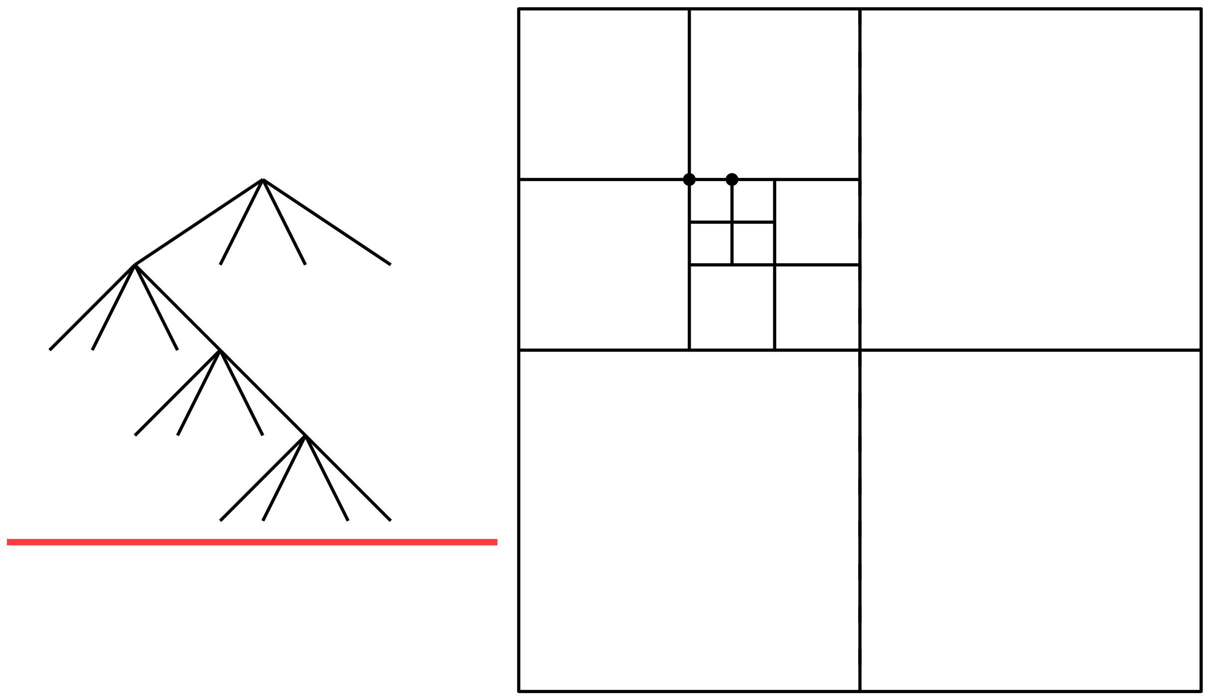


1. Depth not bounded

Solution

1. Snap points to a grid



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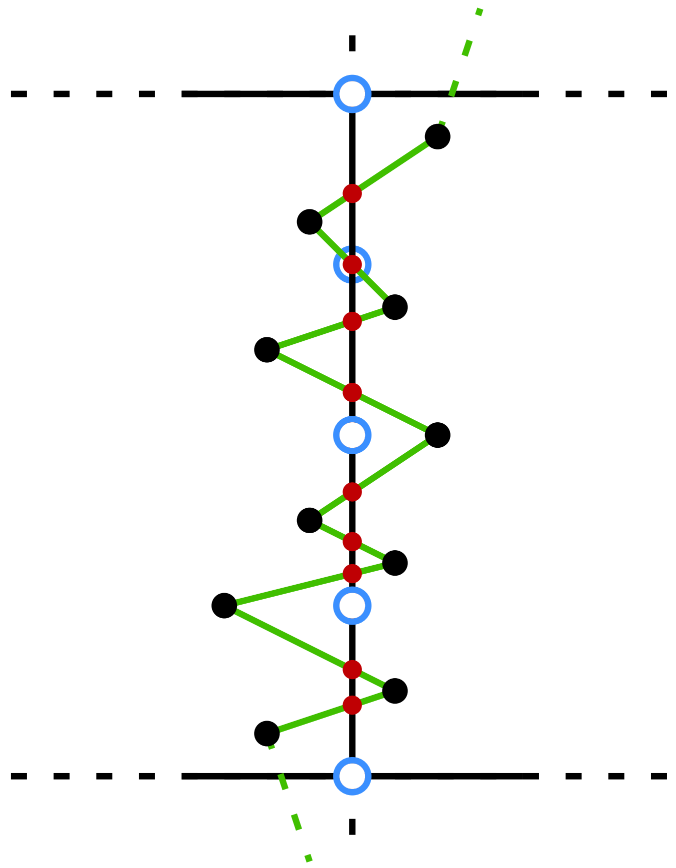

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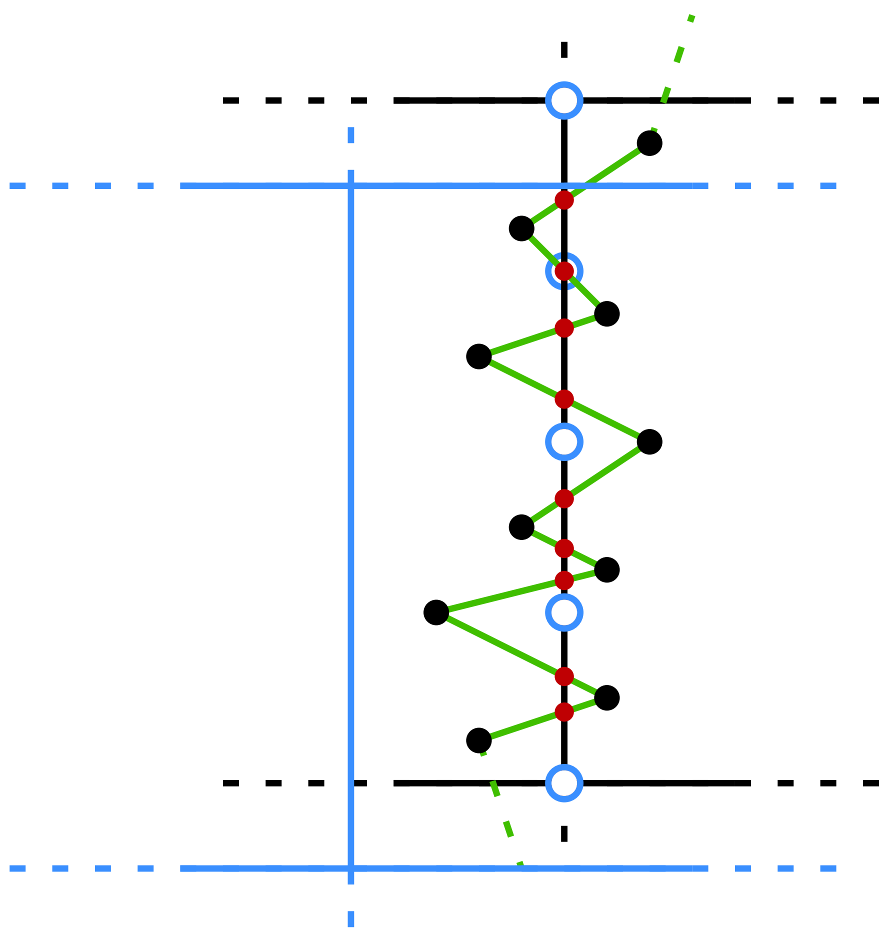
Solution

1. Snap points to a grid

2. Randomize the position of the initial square



A Good Quadtree - Grid

Restrictions on the problem

- P is contained in $[\frac{1}{2}, 1]^2$ and has diameter at least $\frac{1}{4}$
- $\frac{1}{n} < \varepsilon < 1$ the approximation factor where $n = |P|$

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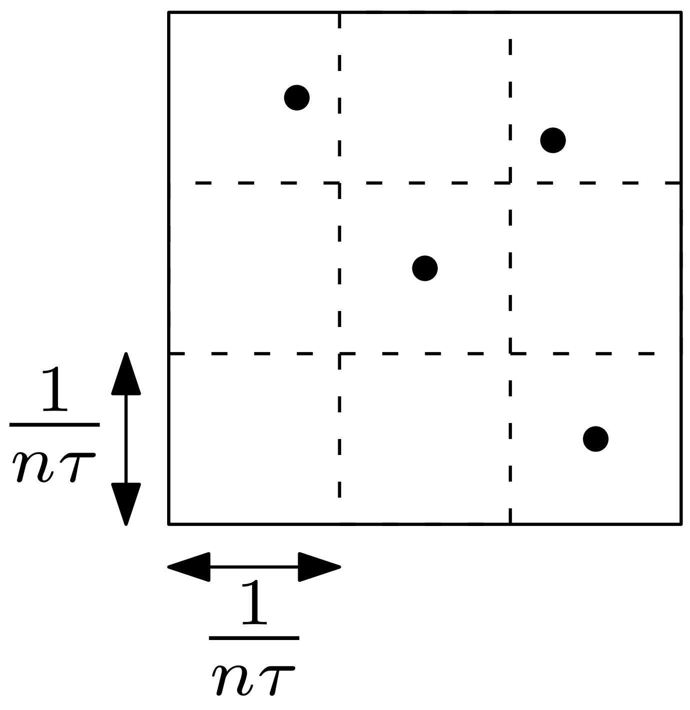
Grid of cells of width $\frac{1}{n\tau}$ where $\tau = \lceil \frac{32}{\varepsilon} \rceil$



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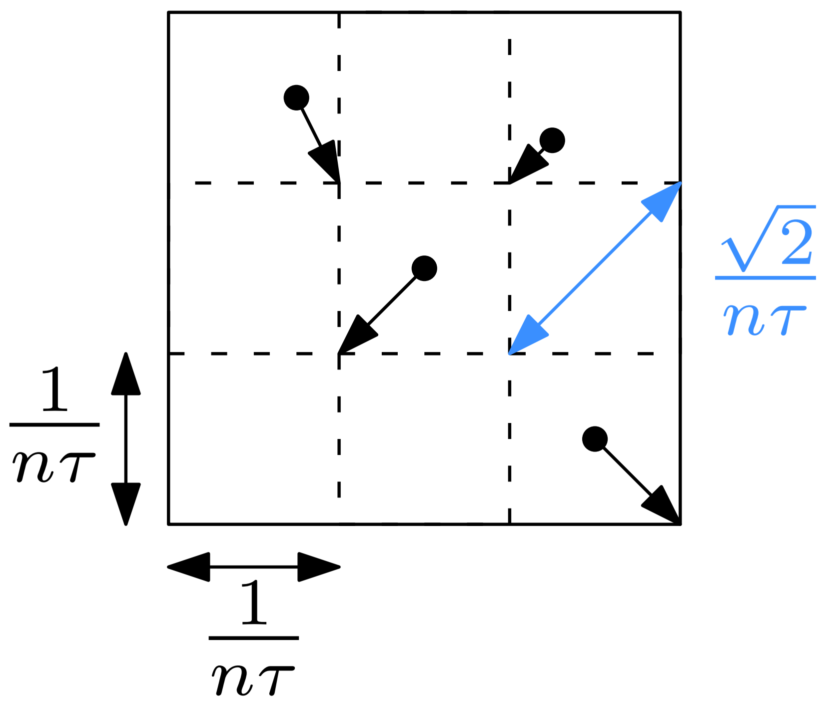
Each point was moved at most $\frac{\sqrt{2}}{2n\tau}$



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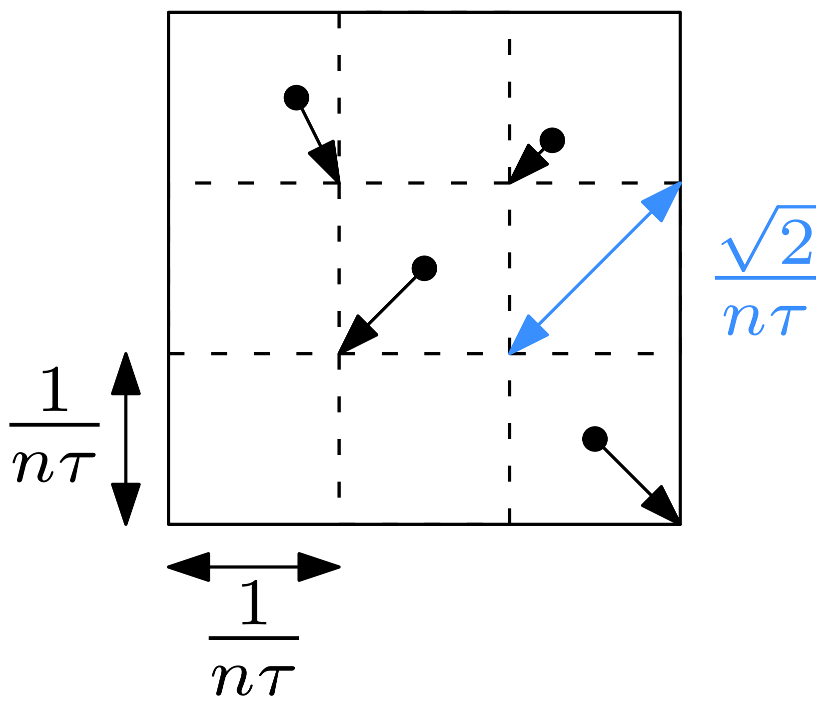
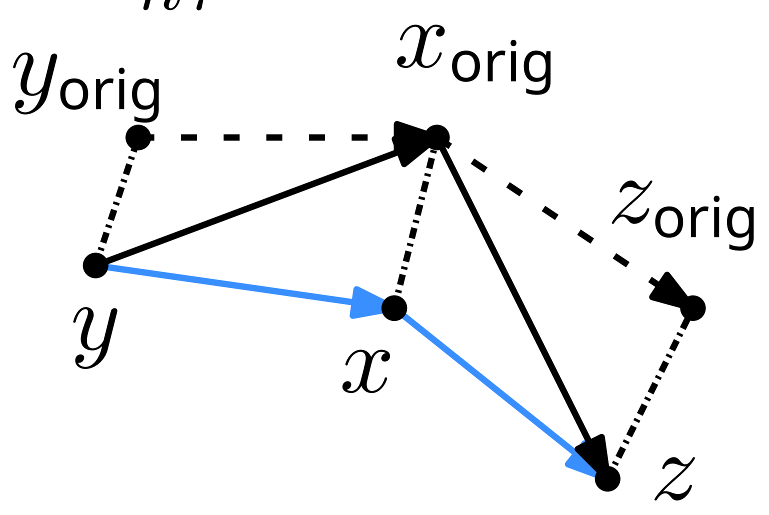
Additional cost: $\leq 2n \cdot \frac{\sqrt{2}}{2n\tau}$





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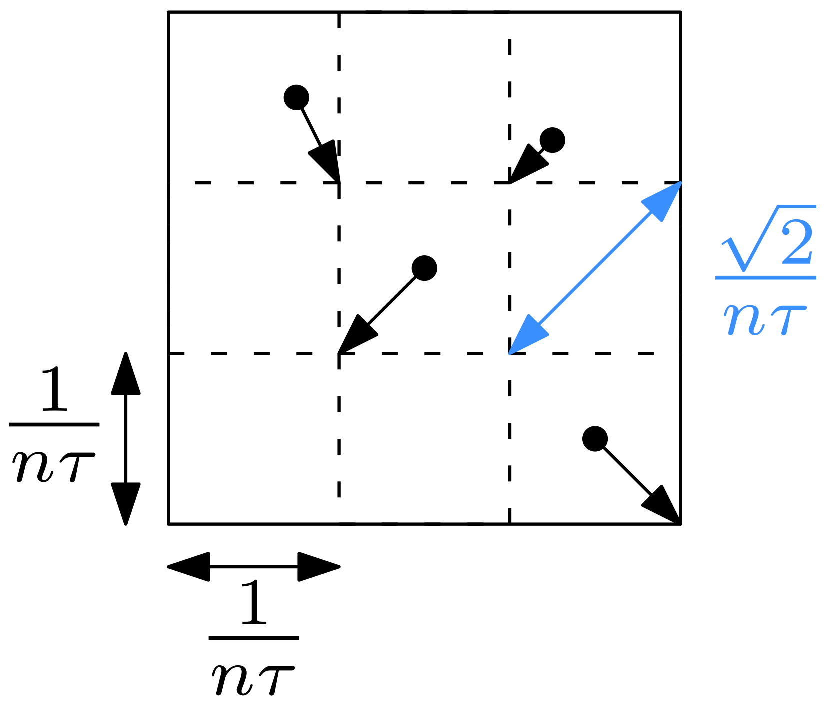

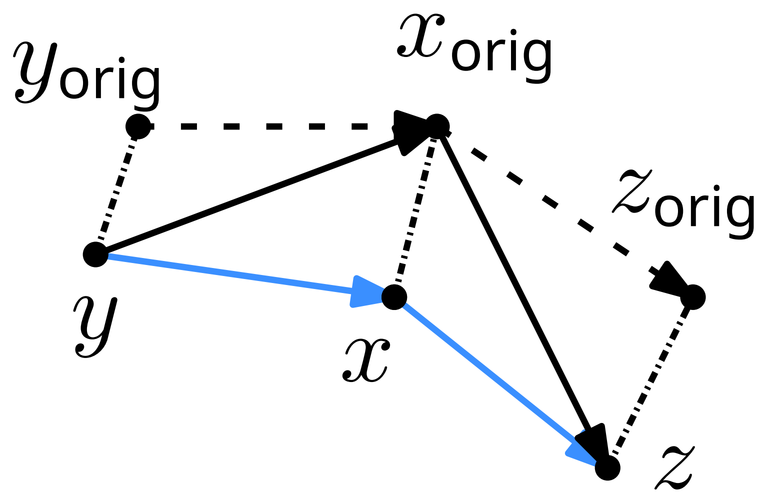

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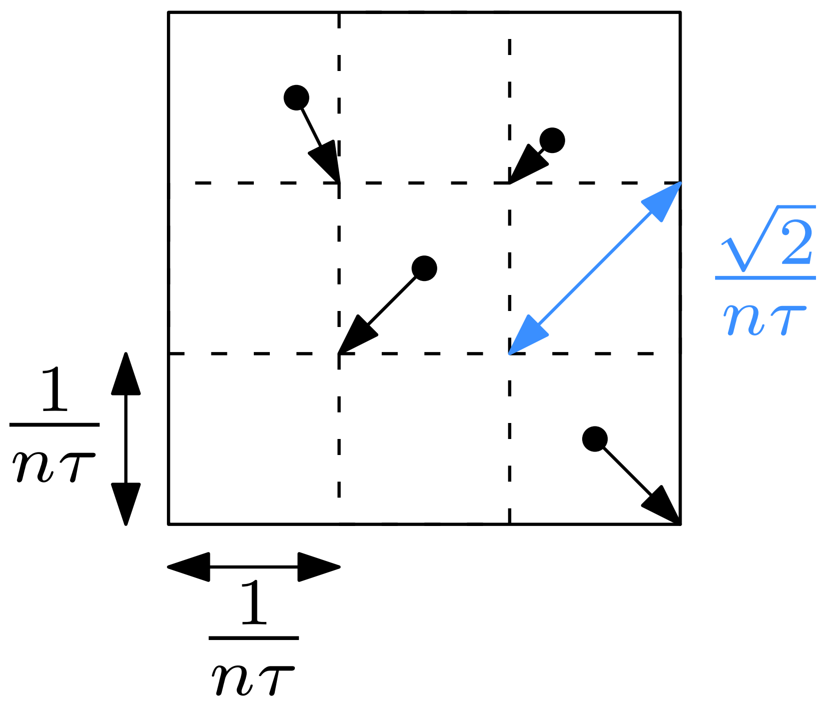

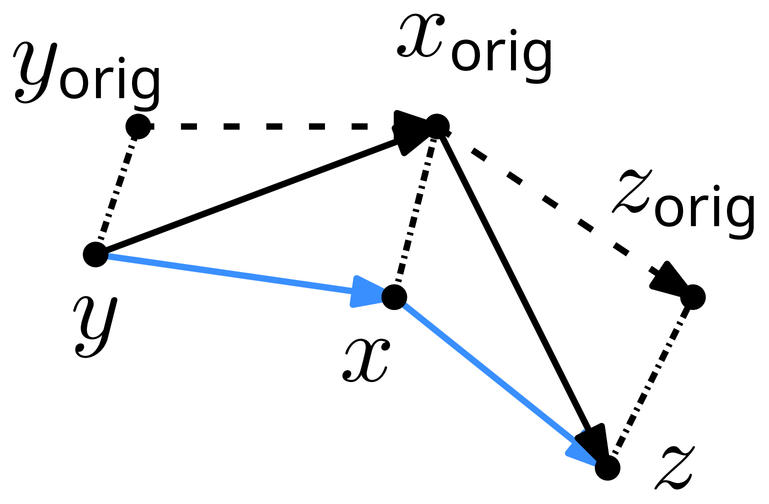

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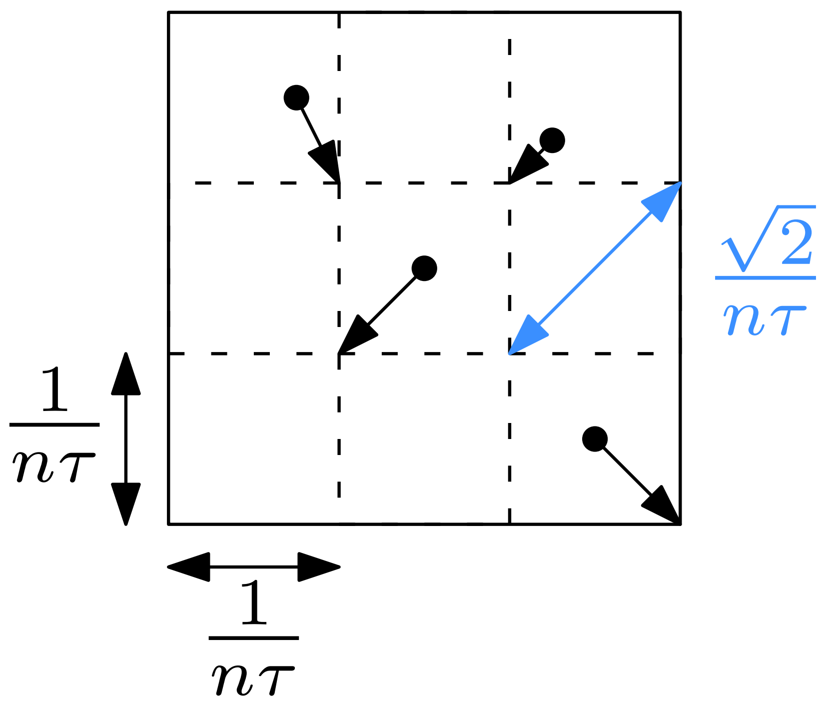
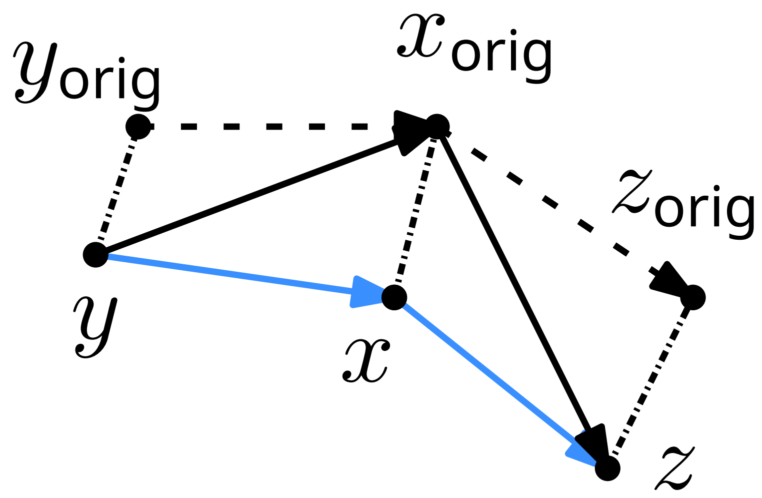
Diameter of P is at least $\frac{1}{4}$, so an optimal solution must be at least as large





A Good Quadtree - Height

Recall: Q contained in $[\frac{1}{2}, 1]^2$ and on grid of width $\frac{1}{n\tau}$

spread: $\Phi(Q) = \frac{\max_{p,q \in Q} ||p-q||}{\min_{p,q \in Q} ||p-q||} = \frac{\sqrt{2}/2}{1/(n\tau)} = \frac{n\tau\sqrt{2}}{2}$

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Let Q a set of n points in the unit square such that its diameter is at least $\frac{1}{2}$. Let $\mathcal{T}$ the quadtree of Q constructed over the unit square. The depth of $\mathcal{T}$ is bounded by $\mathcal{O}(\log \Phi(Q))$, can be constructed in $\mathcal{O}(n \log \Phi(Q))$ time, and is of total size $\mathcal{O}(n \log \Phi(Q))$.

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Height $H = \mathcal{O}\left(\log \frac{n\tau\sqrt{2}}{2}\right) = \mathcal{O}\left(\log \frac{n}{\varepsilon}\right) = \mathcal{O}(\log n)$

Similarly running time and storage of $\mathcal{O}(n \log n)$ follow

Algorithm

`Initialization`(Q):

Construct quadtree $\mathcal{T}$ over Q with height H

Let $k = \frac{90}{\varepsilon} = \mathcal{O}(\frac{1}{\varepsilon})$, $\qquad m \geq \frac{20H}{\varepsilon} = \mathcal{O}(\varepsilon^{-1} \log n)$

`Recursive`(S, M):

1. **if** $|Q_S| = \mathcal{O}(\frac{1}{\varepsilon})$ **then return** `BruteForce`(S, M)
2. $\min_{\text{length}} \leftarrow \infty$
3. **for each** combination $C = [C_1, C_2, C_3, C_4]$ of subproblems of children at S:
4.   **if** C is valid **then**:
5.   cost $\leftarrow$ ParentConnect$(C, S, M) + \sum_{i=1}^{4}$ `Recursive`(C_i)
6. $\min_{\text{length}} \leftarrow \min(\min_{\text{length}}, \text{cost})$
7. **return** $\min_{\text{length}}$

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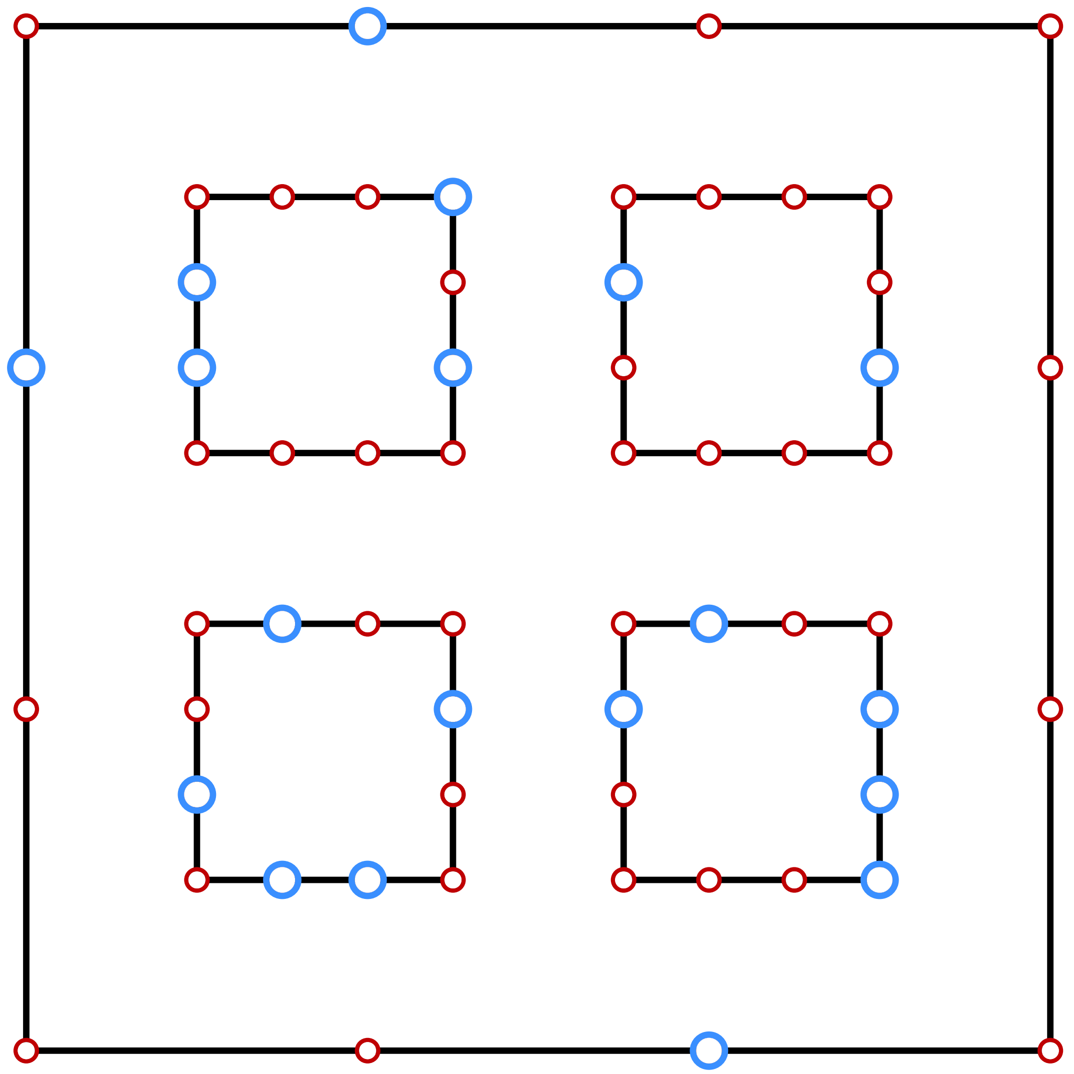
C_i is an arbitrary subproblem for child i

C contains one subproblem for each child



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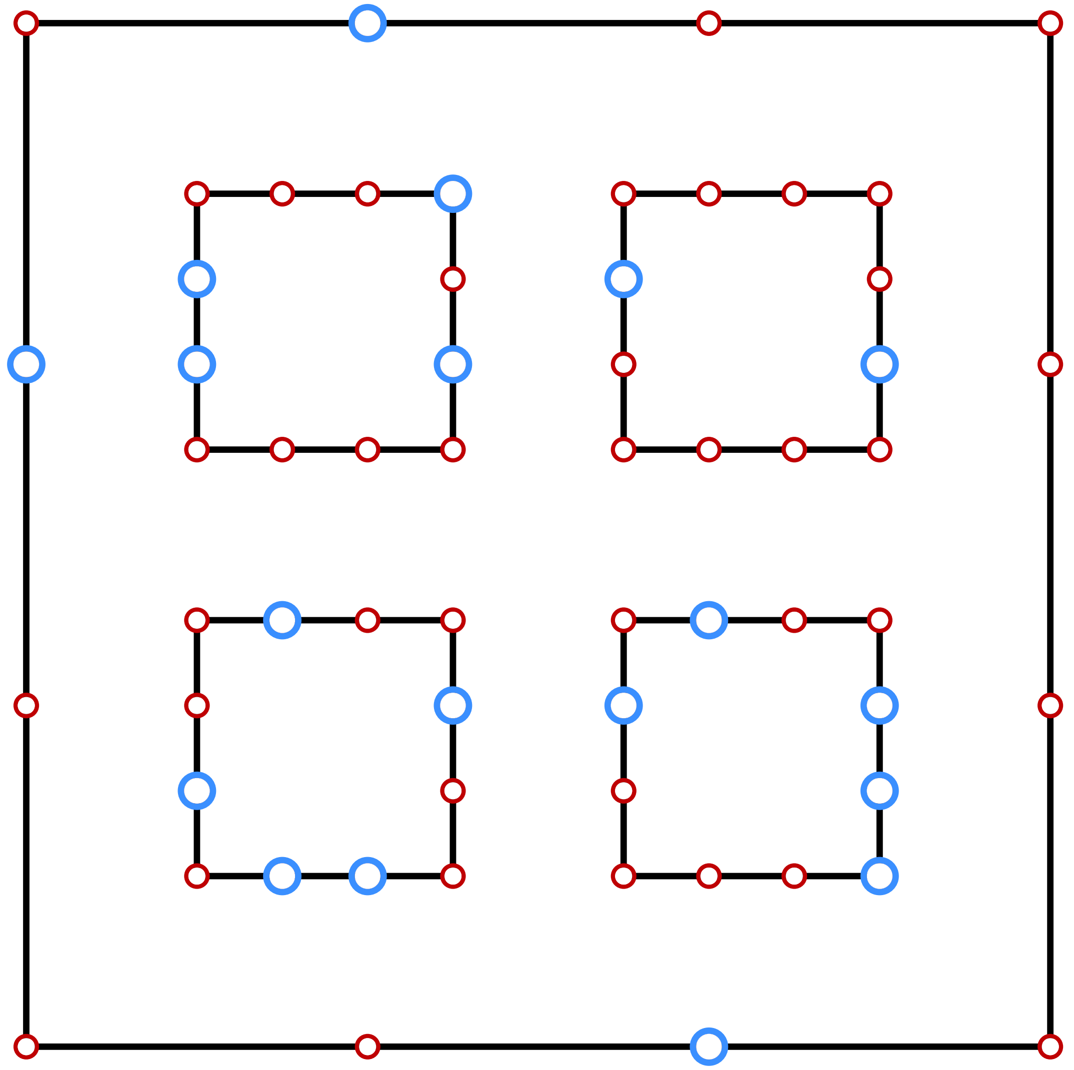

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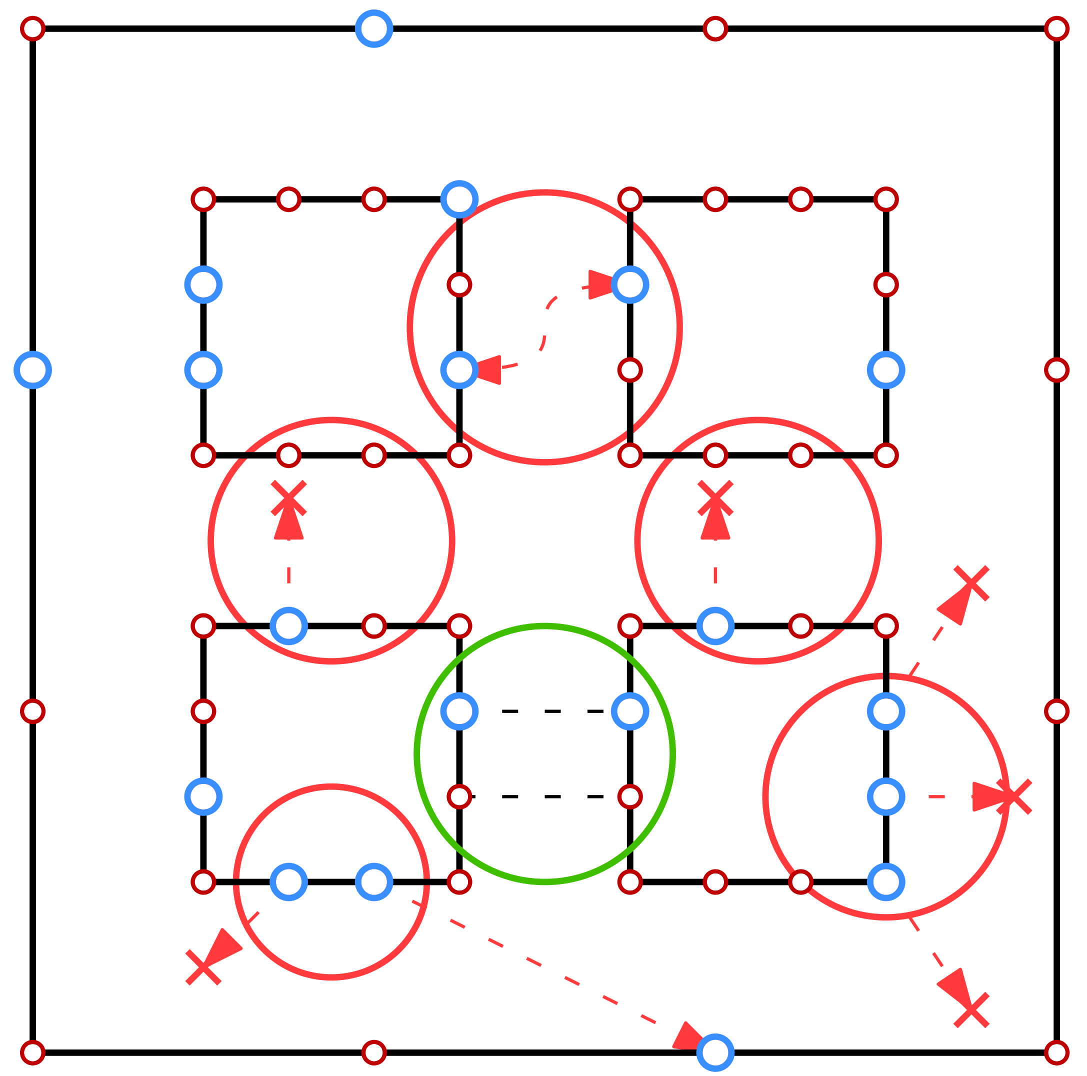

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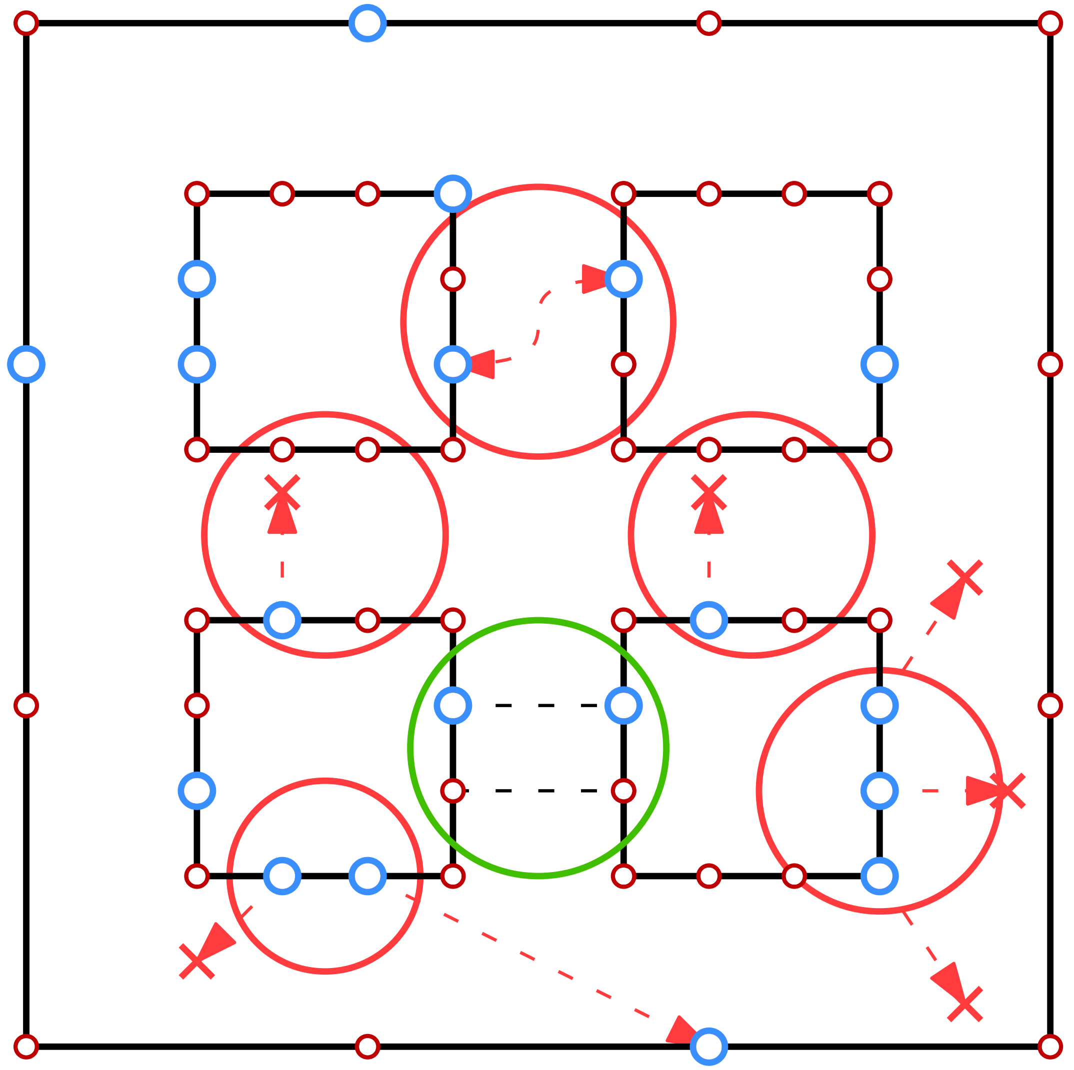

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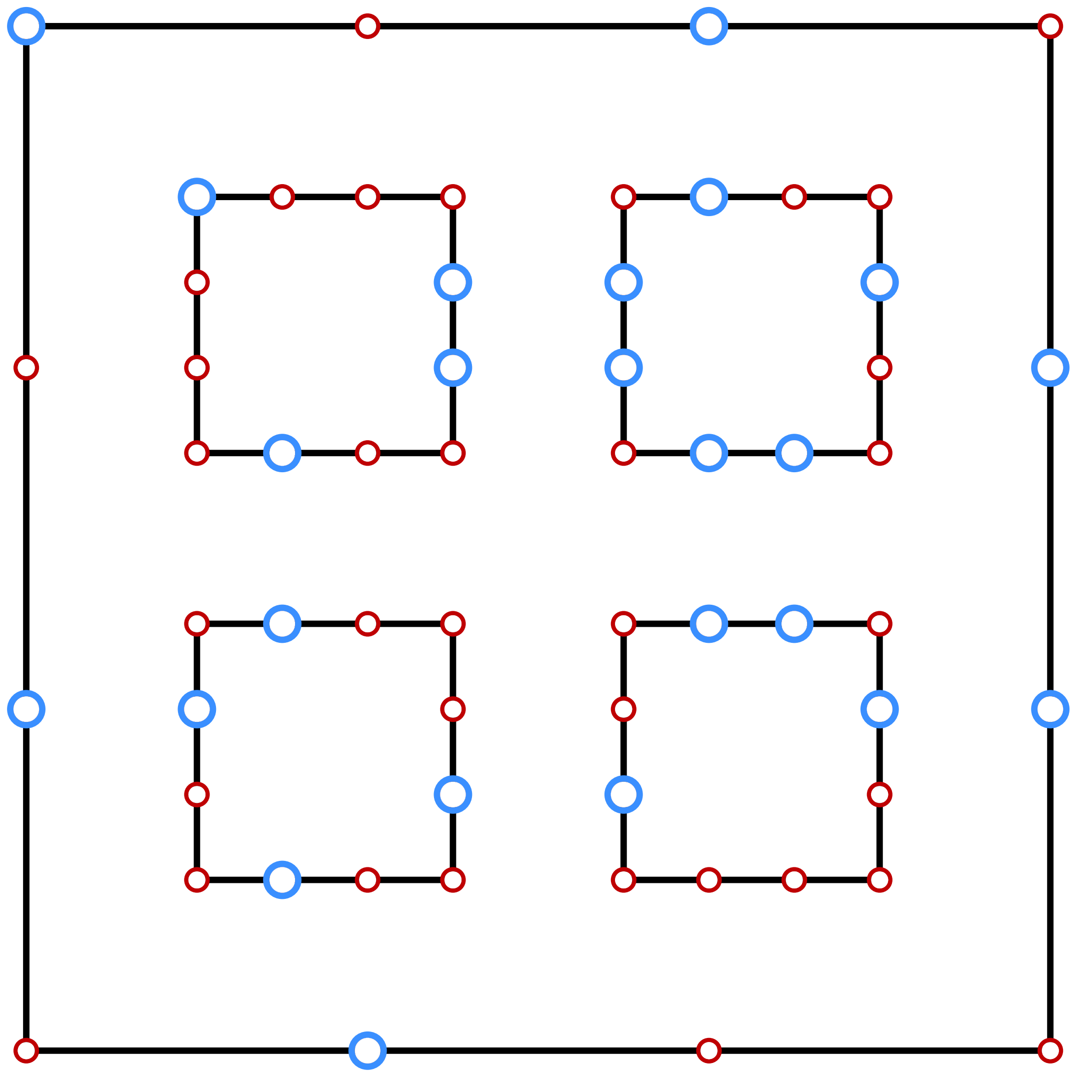

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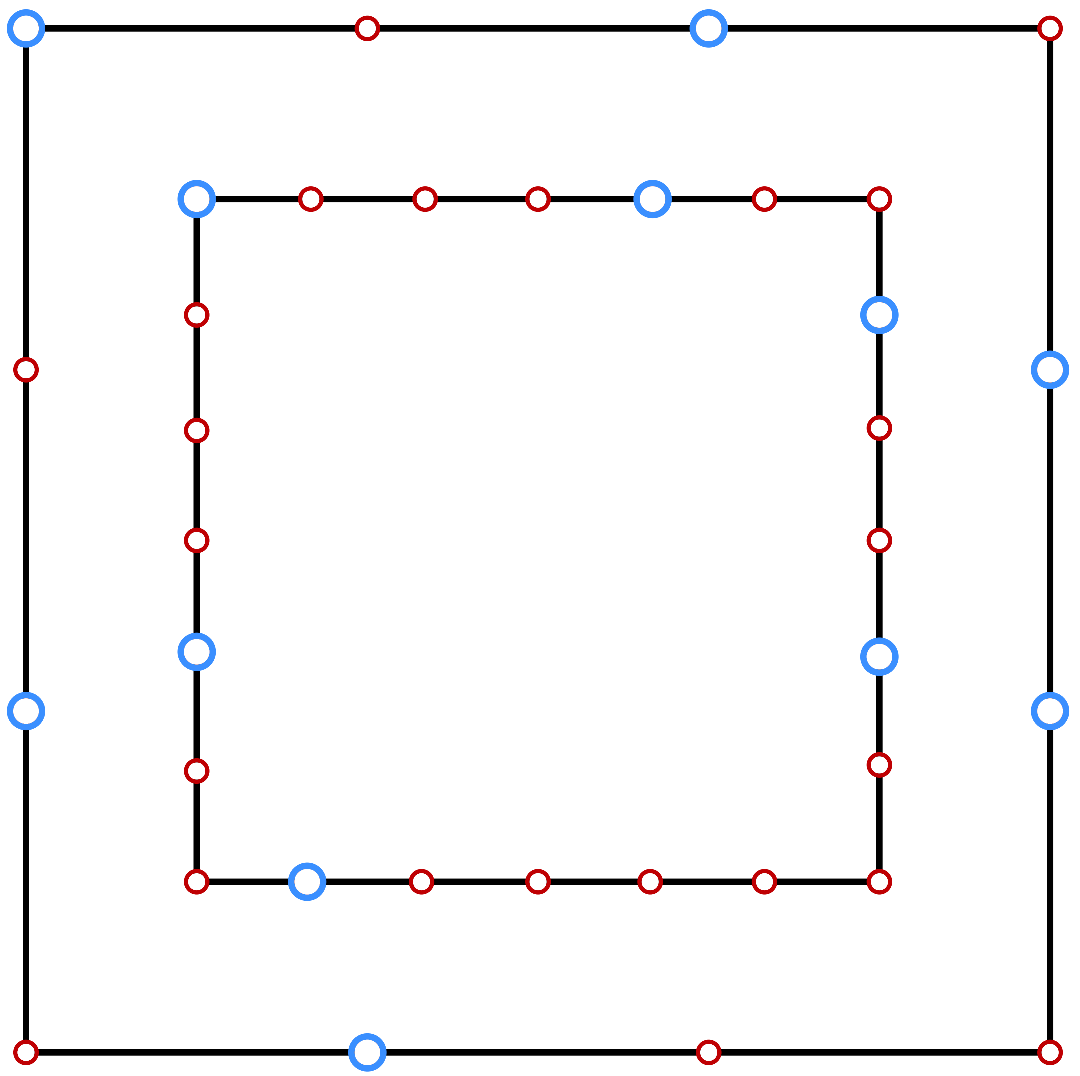

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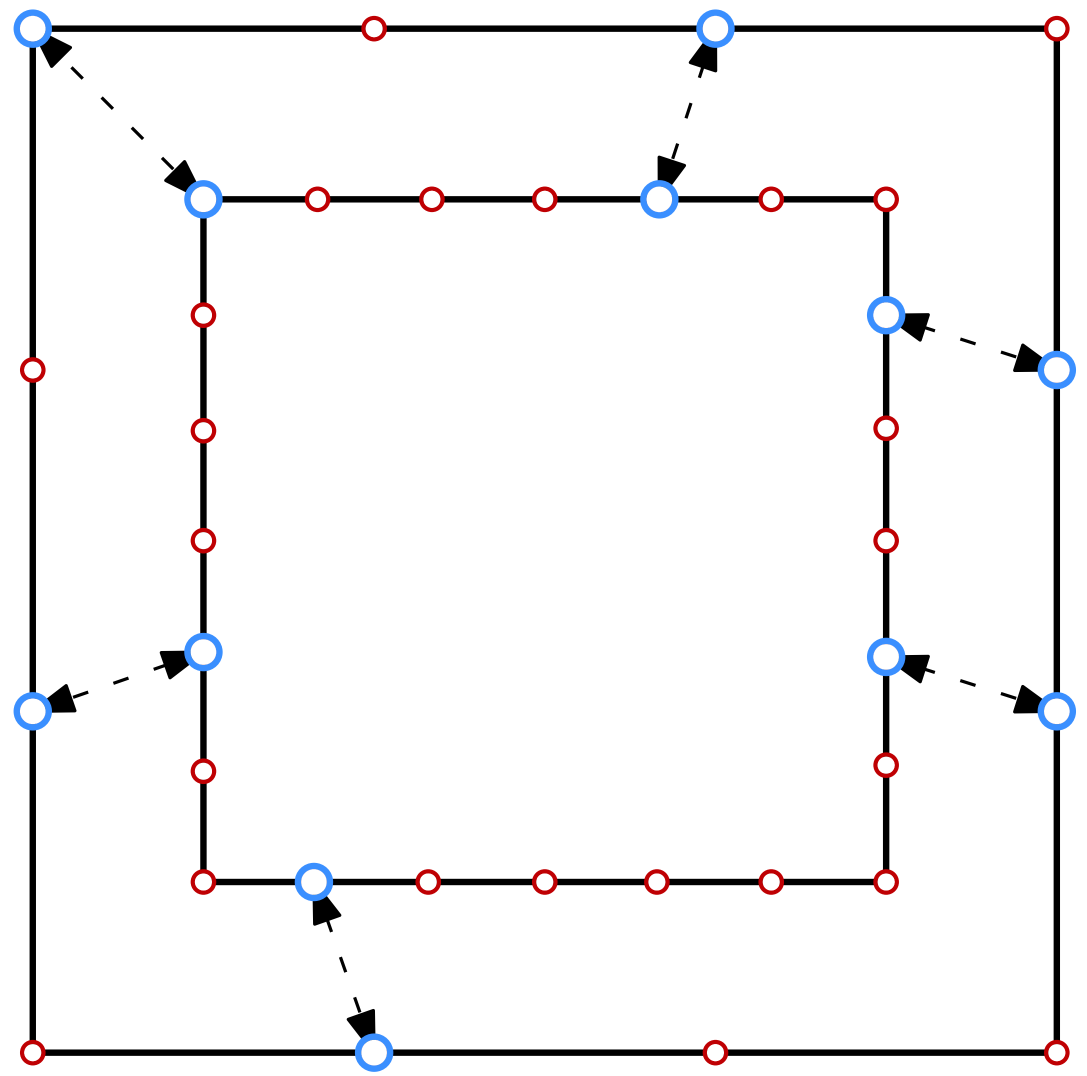

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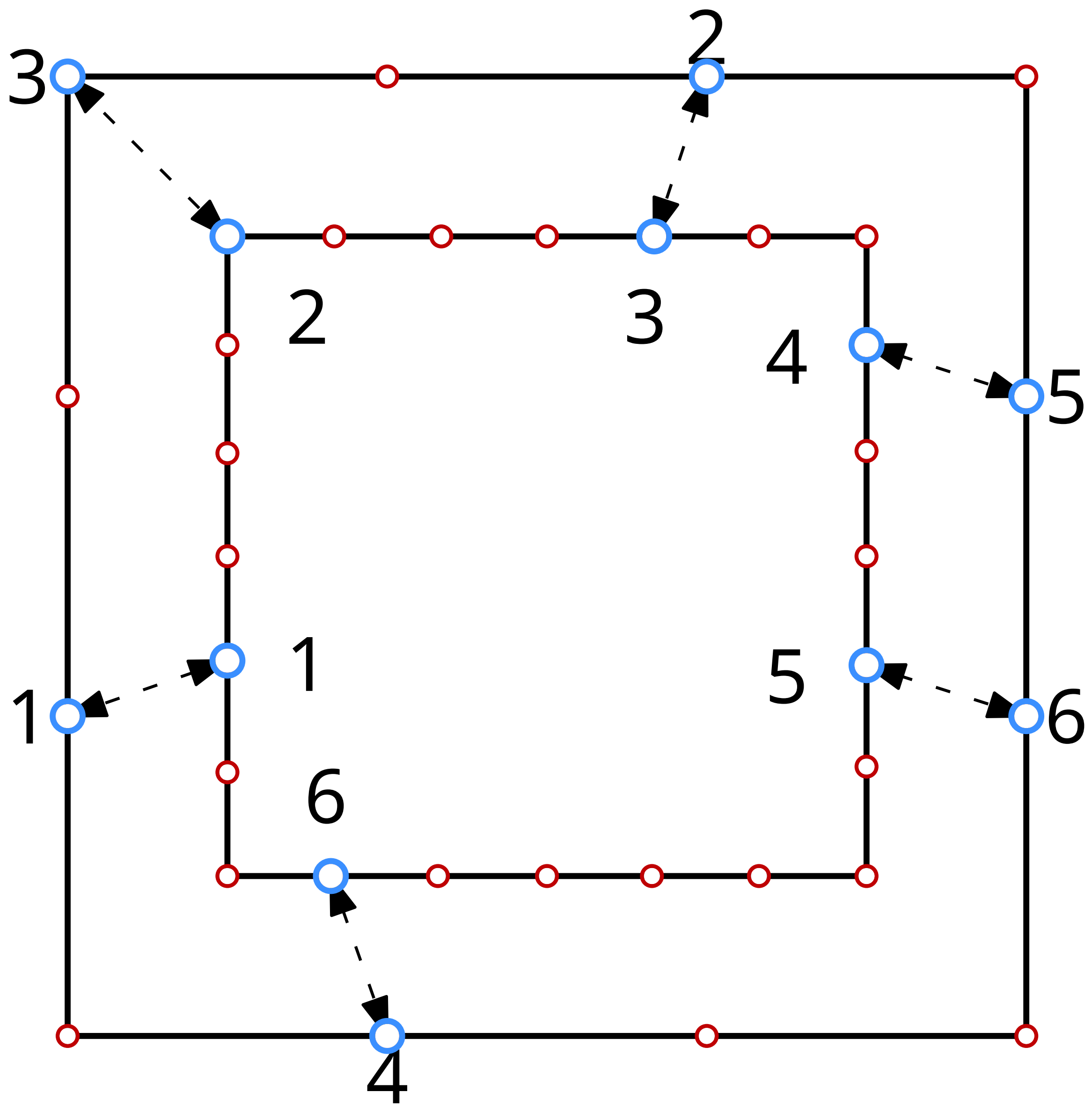
3. **for each** combination $C = [C_1, C_2, C_3, C_4]$ of subproblems of children at S:

4. **if** C is valid **then**:

C_i is an arbitrary subproblem for child i

C contains one subproblem for each child

Valid: even number of portals per child. Portals between children match and number of portals on outside match with parent portals.



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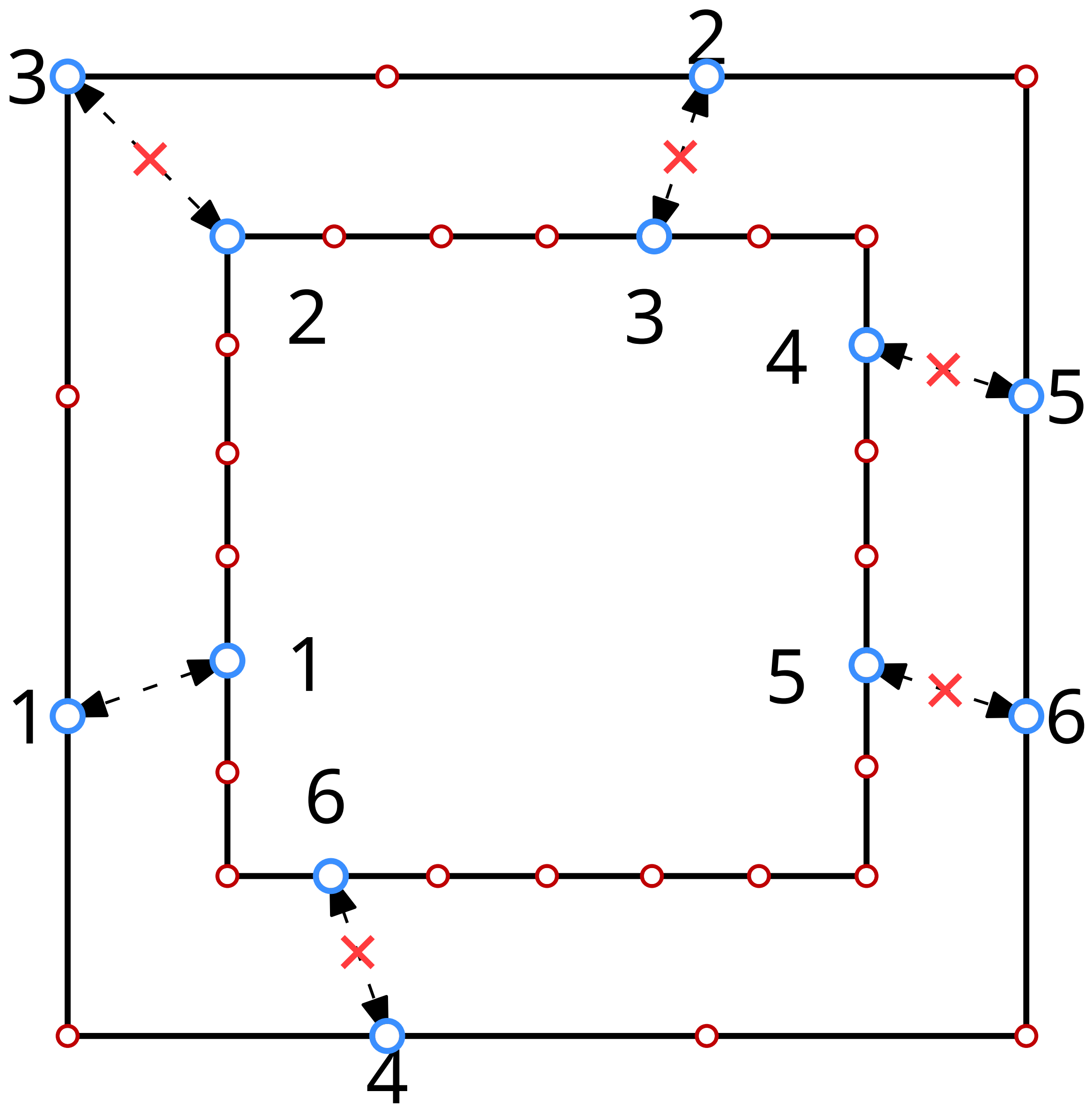

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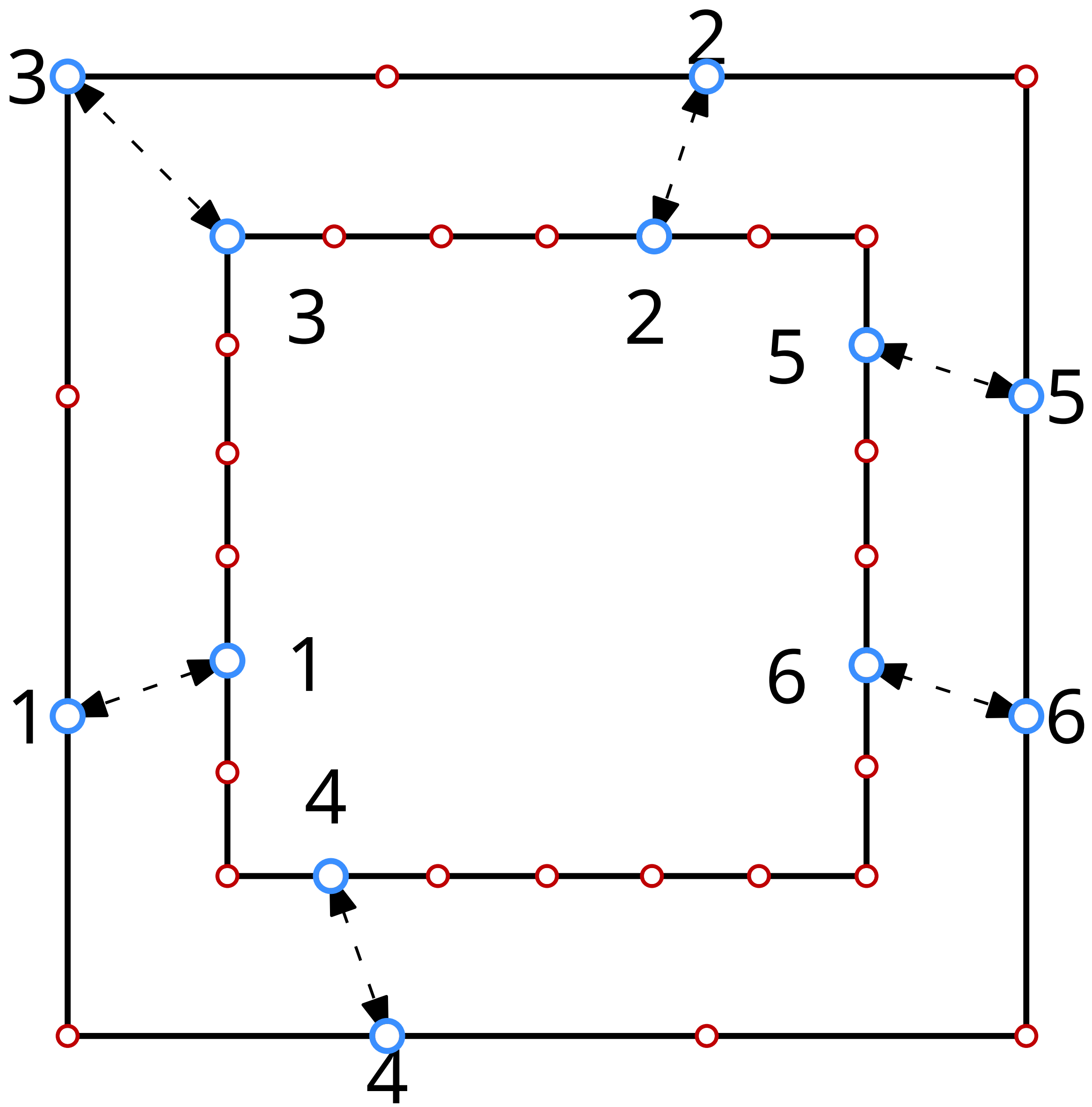

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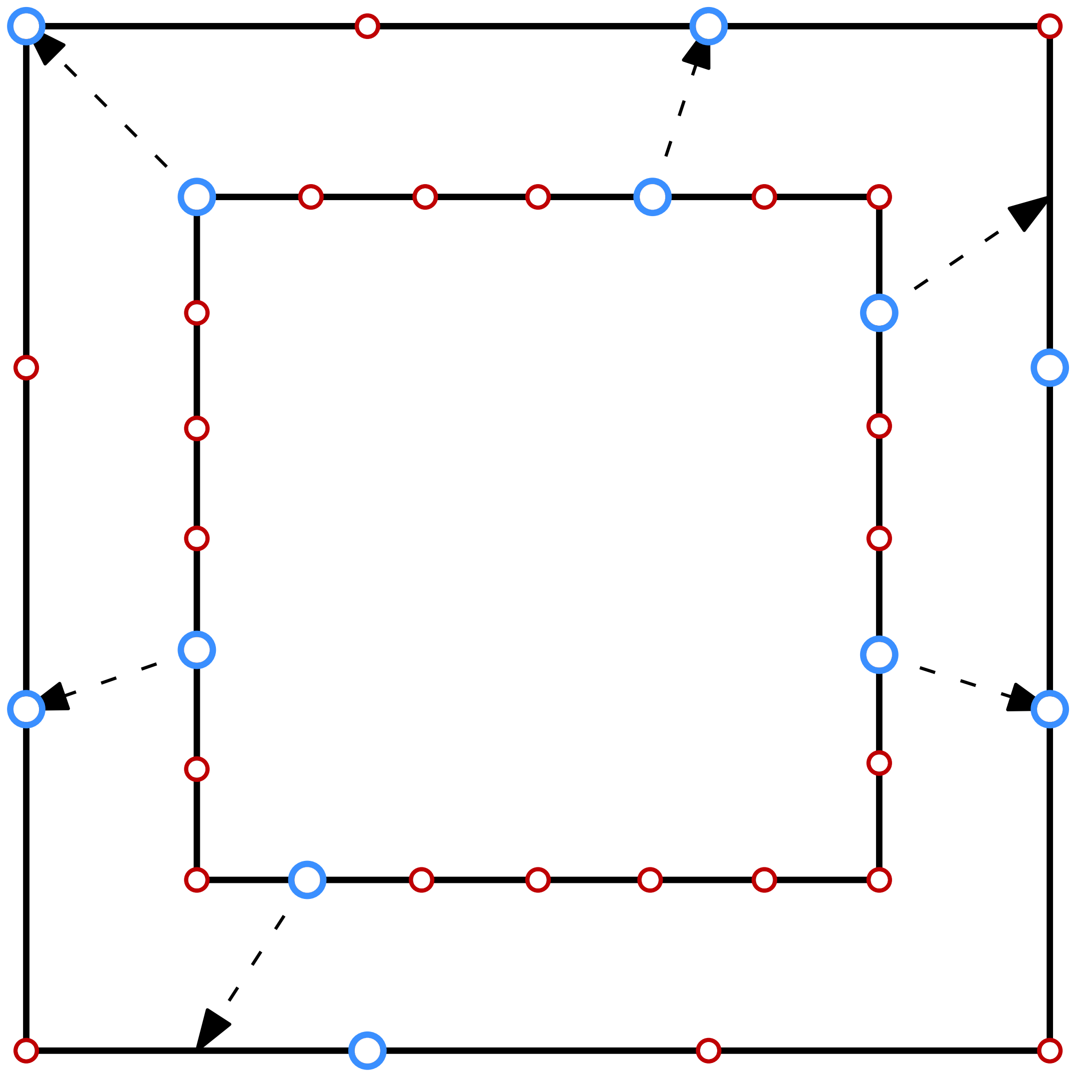

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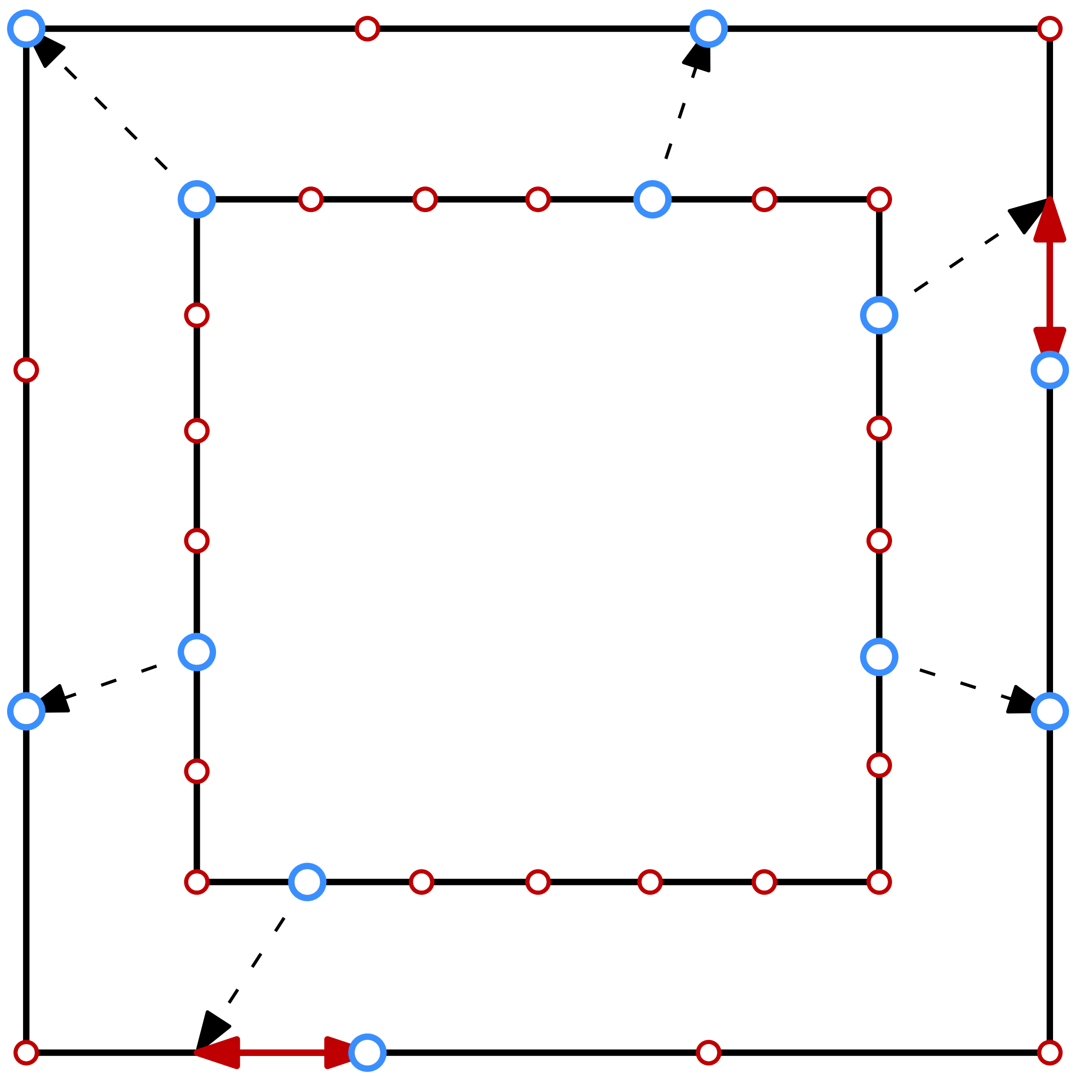
ParentConnect(C, S, M): total misalignment between parent and child portals



Algorithm

`Initialization`(Q):

Construct quadtree $\mathcal{T}$ over Q with height H

Let $k = \frac{90}{\varepsilon} = \mathcal{O}(\frac{1}{\varepsilon})$, $\qquad m \geq \frac{20H}{\varepsilon} = \mathcal{O}(\varepsilon^{-1} \log n)$

`Recursive`(S, M):

1. **if** $|Q_S| = \mathcal{O}(\frac{1}{\varepsilon})$ **then return** `BruteForce`(S, M)
2. $\min_{\text{length}} \leftarrow \infty$
3. **for each** combination $C = [C_1, C_2, C_3, C_4]$ of subproblems of children at S:
4.   **if** C is valid **then**:
5.   cost $\leftarrow$ ParentConnect$(C, S, M) + \sum_{i=1}^{4}$ `Recursive`(C_i)
6. $\min_{\text{length}} \leftarrow \min(\min_{\text{length}}, \text{cost})$
7. **return** $\min_{\text{length}}$

Use memoization to make it a DP algorithm

Overview

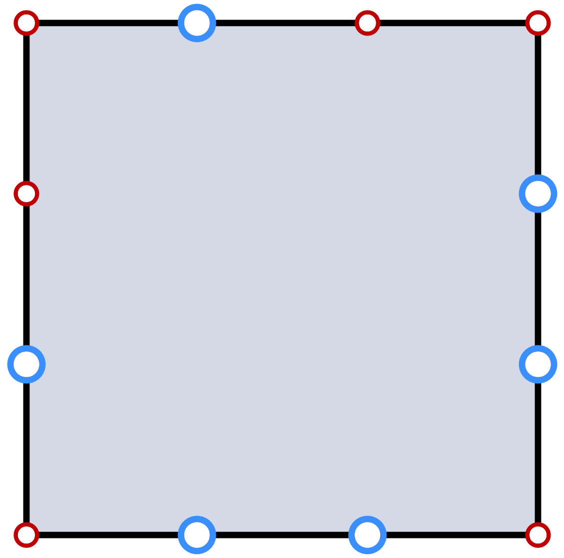
1. Intuition ✓
2. Subproblems ✓
3. Algorithm ✓
4. Running time
5. Quality of approximation

Running Time - Number of Subproblems per Square

Consider a square S and its portals

It has $4m + 4$ portals on its boundary, each of which is used at most twice

At most k portals are used on each side of S



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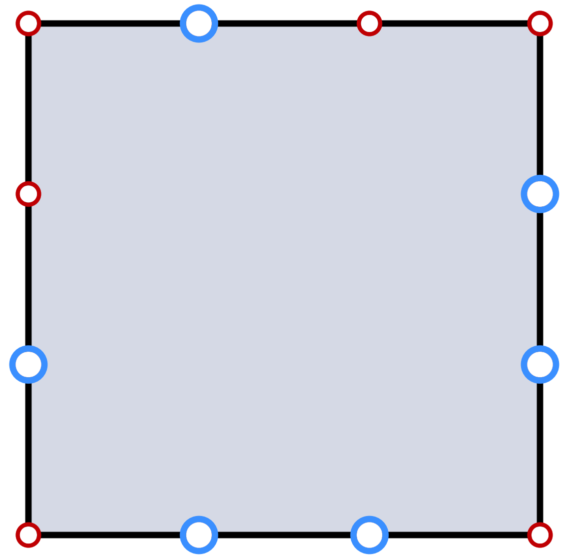
There are $i!$ orderings of these portals



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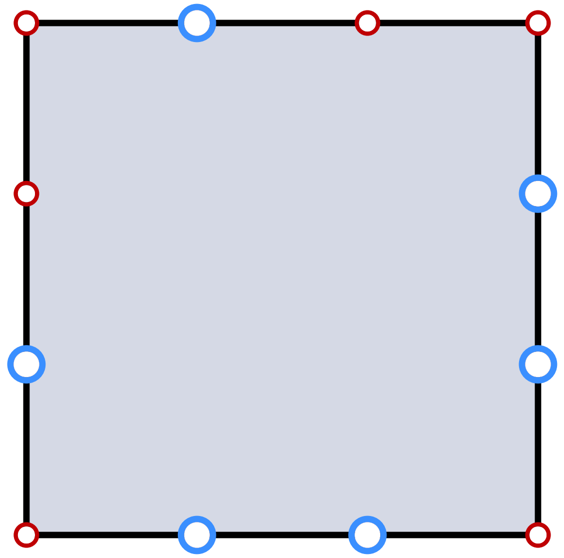
Bound T the number of subproblems with S as square

$$T = \sum_{i=0}^{8k} \binom{8m+8}{i} i!$$



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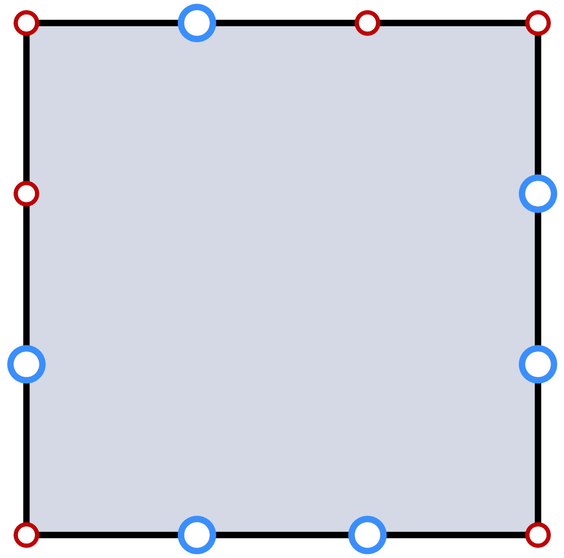

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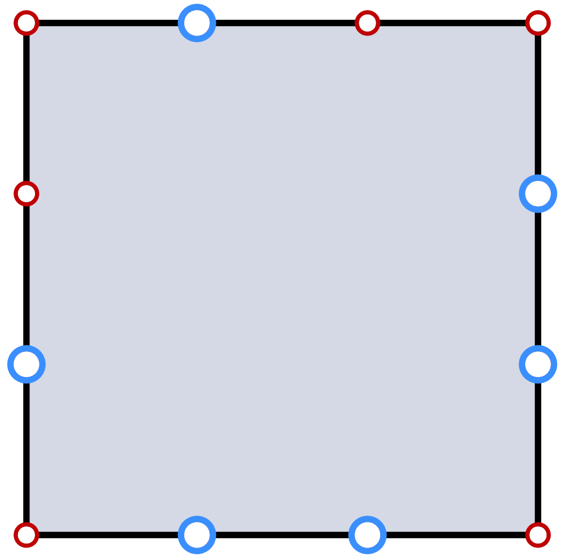

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$$\leq 8k(8m+8)^{8k}$$

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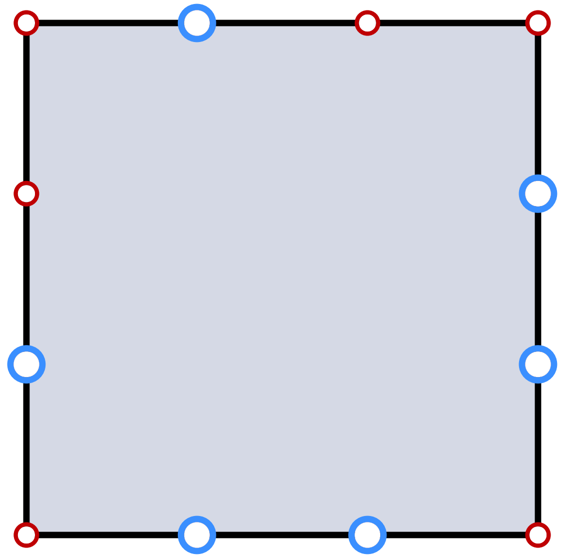
$$\leq 8k(8m+8)^{8k} = (\varepsilon^{-1} \log n)^{\mathcal{O}(1/\varepsilon)}$$



Recall

$$k = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

$$m = \mathcal{O}\left(\frac{\log n}{\varepsilon}\right)$$

Running Time

Recall: quadtree had $\mathcal{O}(n \log n)$ nodes and one square per node
Claim: brute force on a square with $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$ points (base case) can be done in $T = (\varepsilon^{-1} \log n)^{\mathcal{O}(1/\varepsilon)}$ time

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Algorithm considers all combinations of subproblems for the squares of the children for each node

The number of subproblems per child is at most T
The number of combinations for four children is T^4
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Total running time:
$\mathcal{O}(n \log n + (n \log n)T^5) = \mathcal{O}((n \log n)T^5) = n(\varepsilon^{-1} \log n)^{\mathcal{O}(1/\varepsilon)}$

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Quality of Approximation

Introduced error when:

- snapping to the grid
- bounding the number of intersections at k per side of each square
- requiring the use of portals

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  uses: shifting

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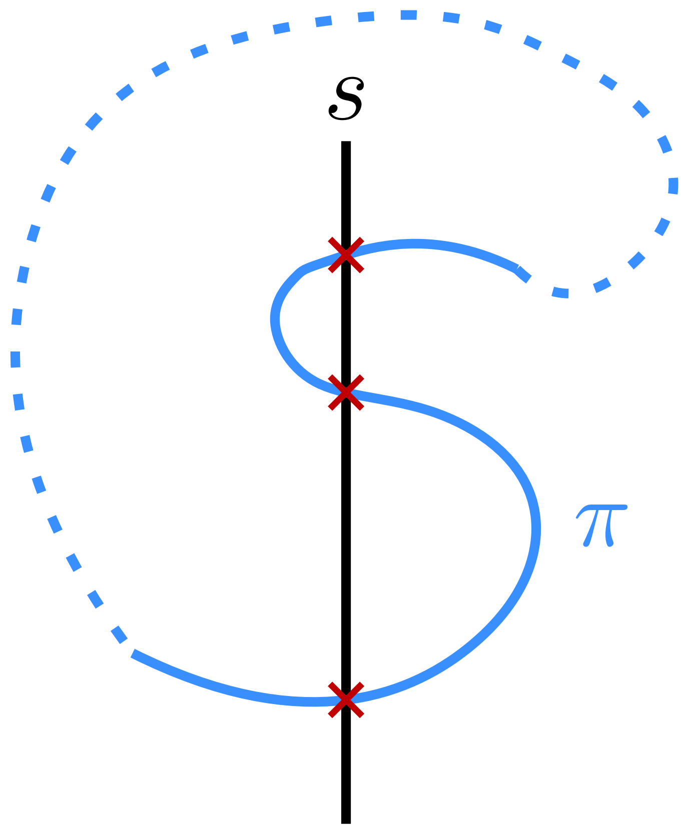
Patching lemma: closed curve π crossing segment s at least $k \geq 3$ times can be replaced by closed curve π' crossing s at most twice such that $||\pi'|| \leq ||\pi|| + 4||s||$



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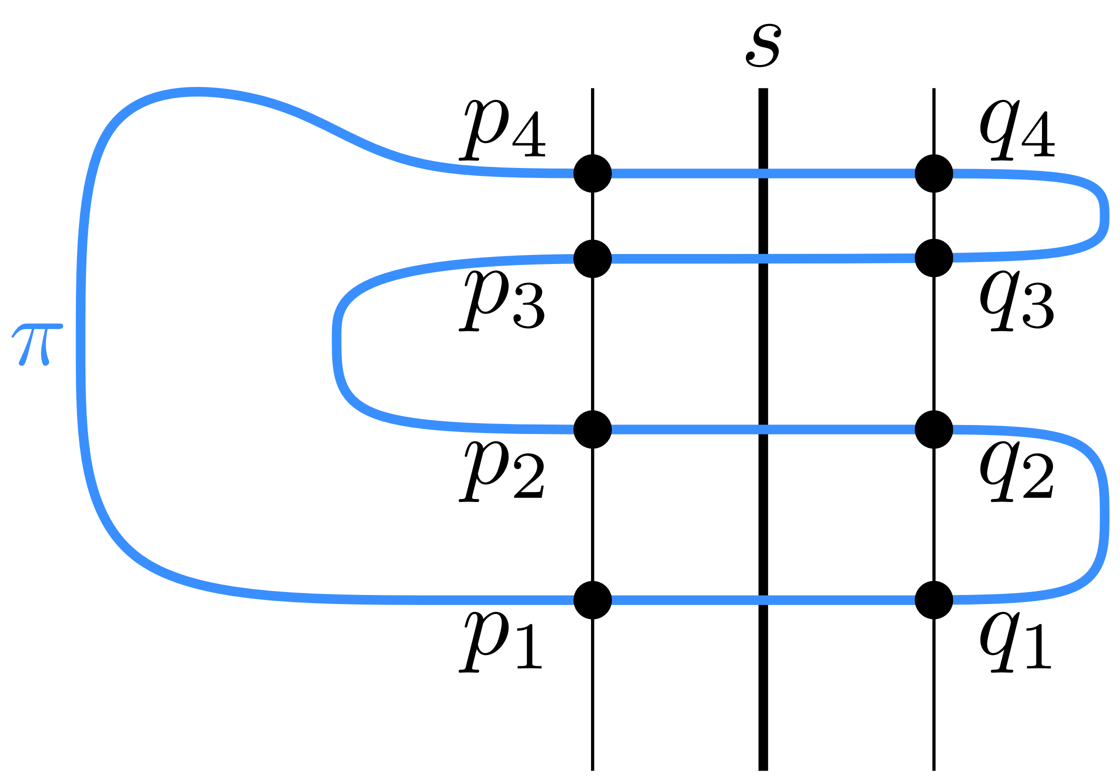

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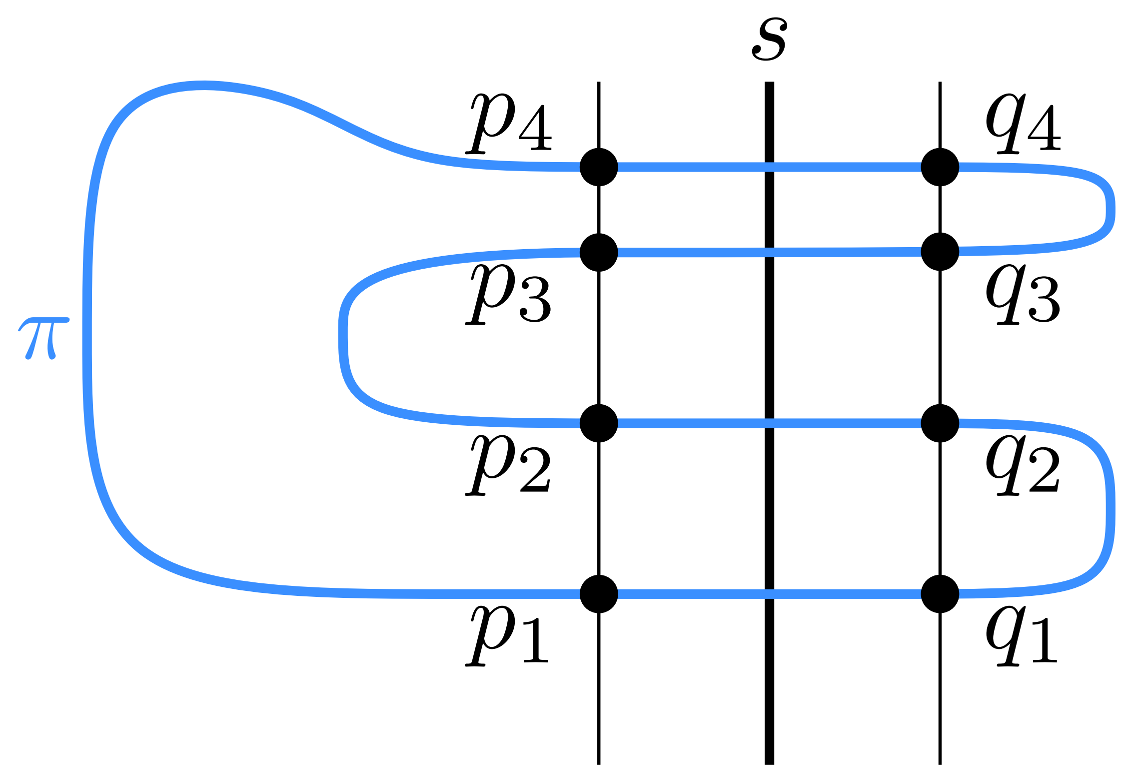
Idea: construct an Eulerian tour including π that crosses s at most twice



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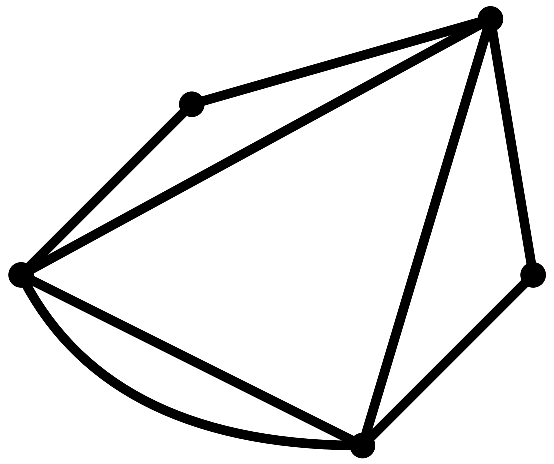
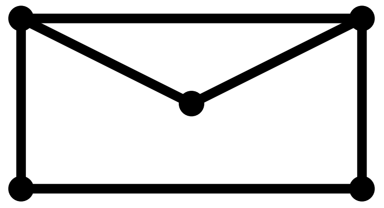
Theorem: a connected graph admits an Eulerian tour if and only if its vertices have even degree





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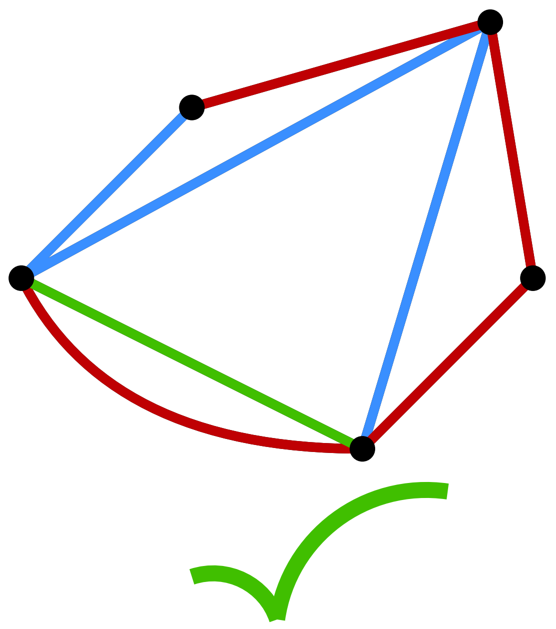

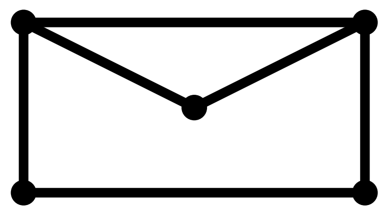

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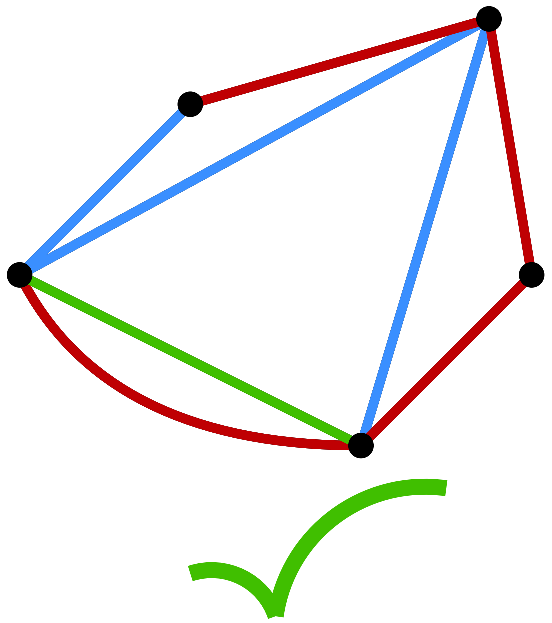

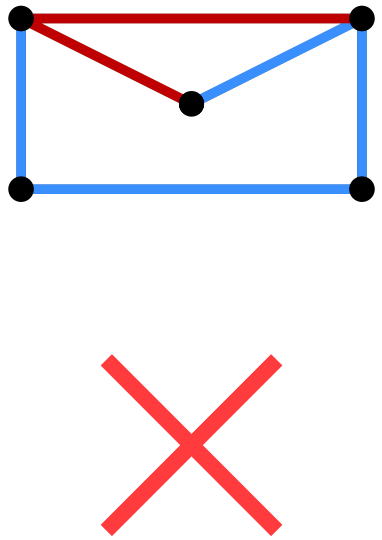

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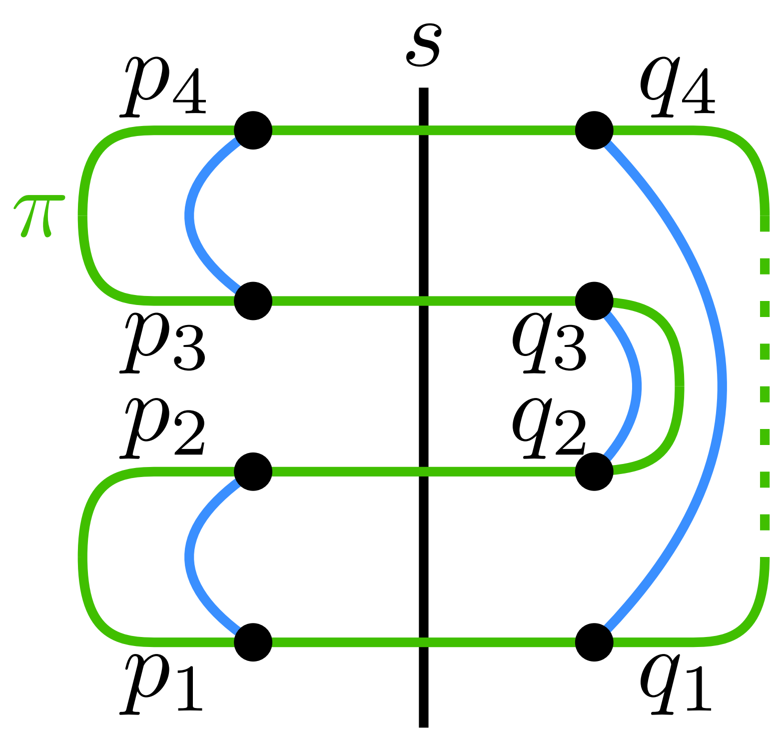

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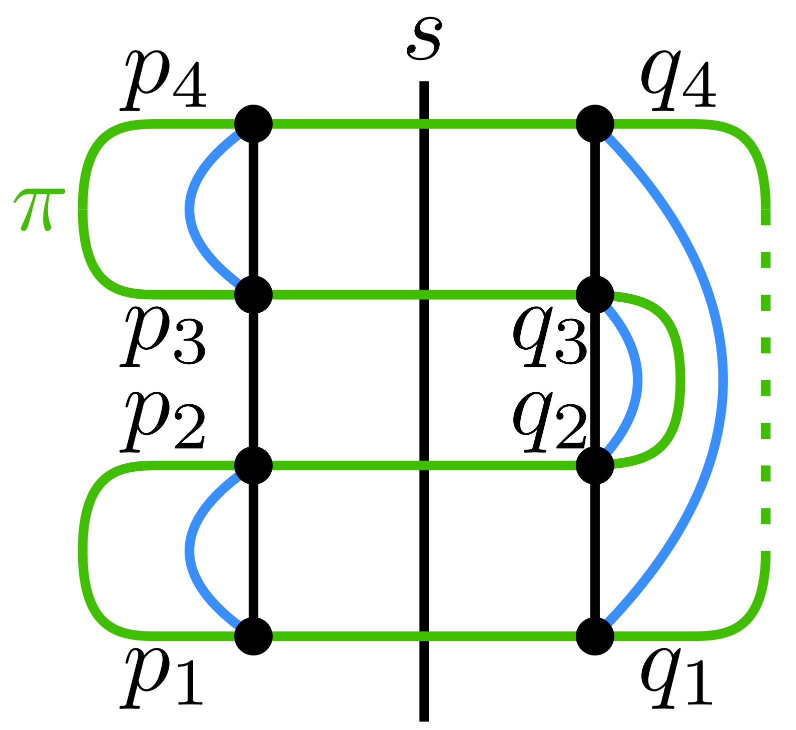

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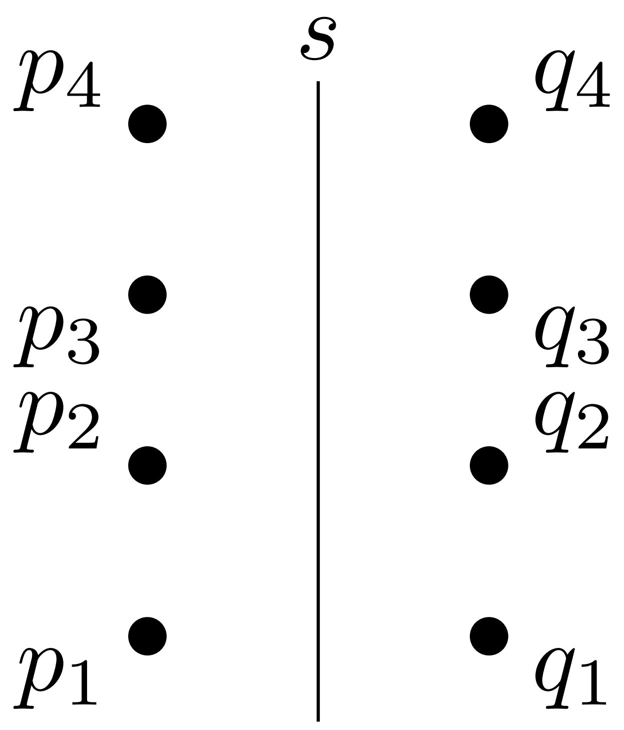

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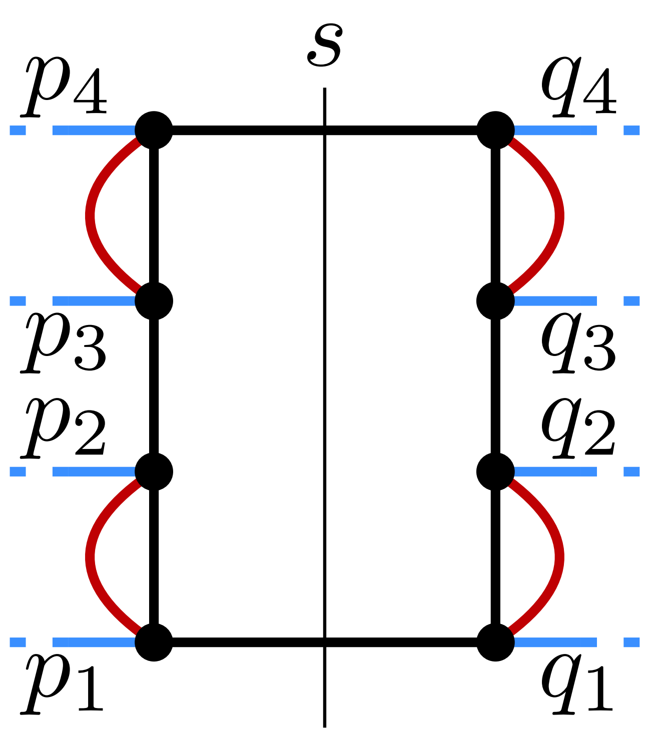
Construct graph $G = (V, E)$ with $V = \bigcup_{i=1}^{k} \{p_i, q_i\}$

$E = \bigcup_{i=1}^{k-1} \{s_i, s_i'\} \cup E_\pi \cup \bigcup_{i=1 (\text{odd})}^{k-1} \{s_i, s_i'\} \cup \{(p_1, q_1)\}$

and also add (p_k, q_k) if k is even



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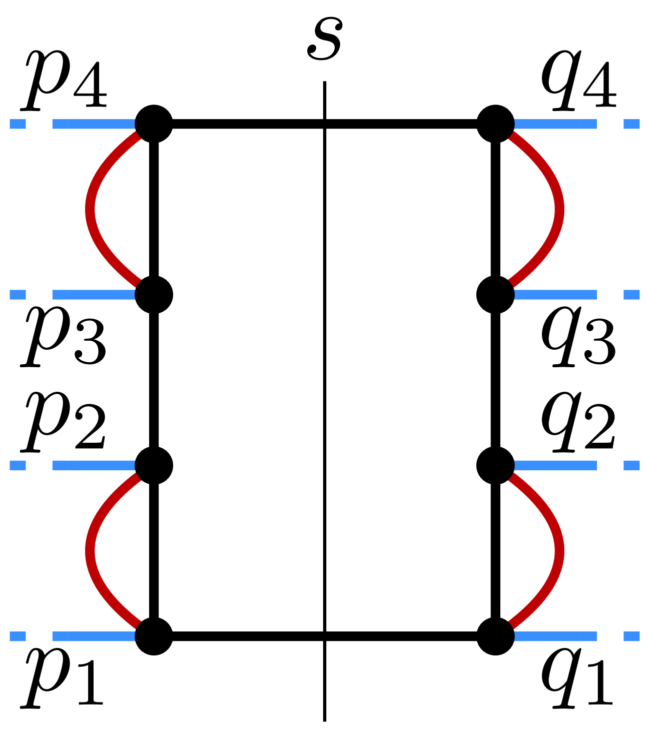
G is connected and its vertices have even degree

Hence it admits an Eulerian tour π'

π' visits all of π outside of s as $E_\pi \subseteq E$

π' crosses s at most twice: at (p_1, q_1) and (p_k, q_k)



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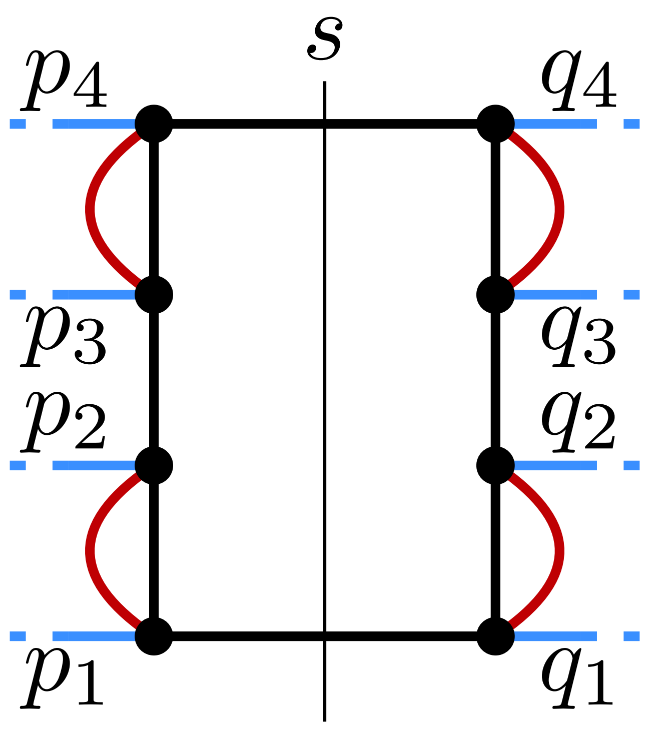

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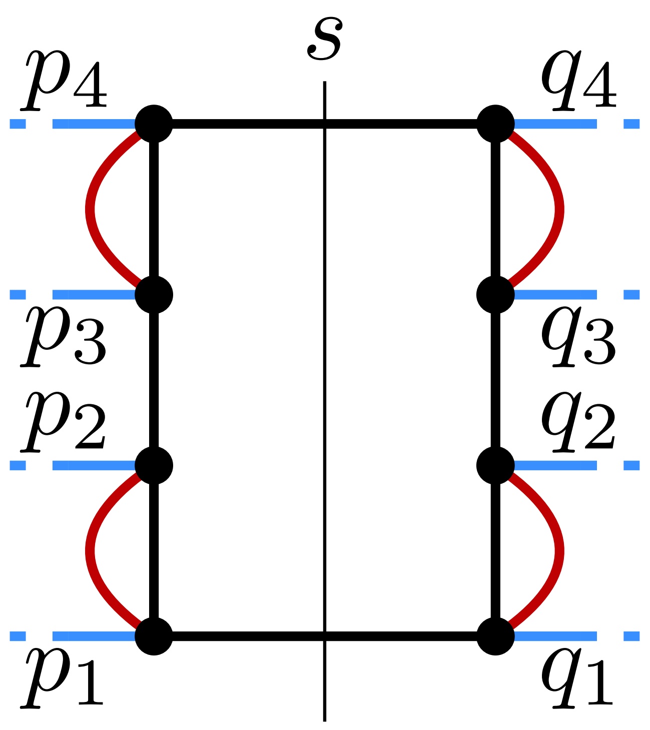
$||\pi'|| \leq ||\pi|| + 2||s||| + 2||s|| = ||\pi|| + 4||s||$



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Using portals only twice

apply lemma to portals $\leftarrow$ segments of length 0

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apply lemma to portals $\leftarrow$ segments of length 0

Using only k portals of a square

bottom-up (i.e. starting with small cells): When $> k$ intersections, patch!

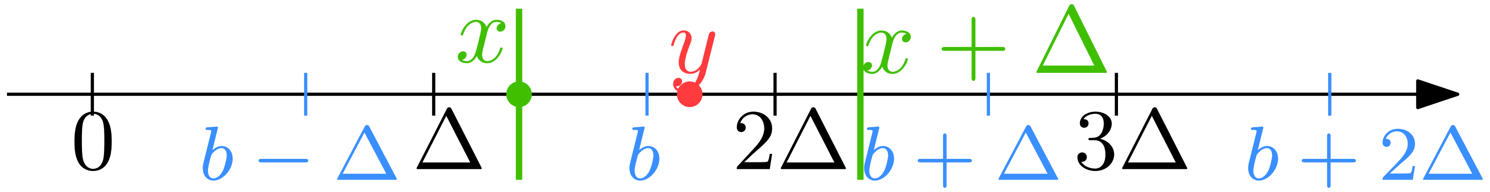
intuition: patching on low levels: relatively cheap, and also helps higher levels

+ fewer (exponentially decreasing) intersections at higher levels: shifting

Shifted Grids

recall: shifted partition of real line

Let $\Delta > 0$ and $b \in [0, \Delta]$ uniformly distributed. We shift the grid G_Δ by b



$$h_{b,\Delta}(x) = \lfloor \frac{x-b}{\Delta} \rfloor$$

Lemma: For $x, y \in \mathbb{R}$ holds $\mathbb{P}\left[h_{b,\Delta}(x) \neq h_{b,\Delta}(y)\right] = min\left(\frac{|x-y|}{\Delta}, 1\right)$

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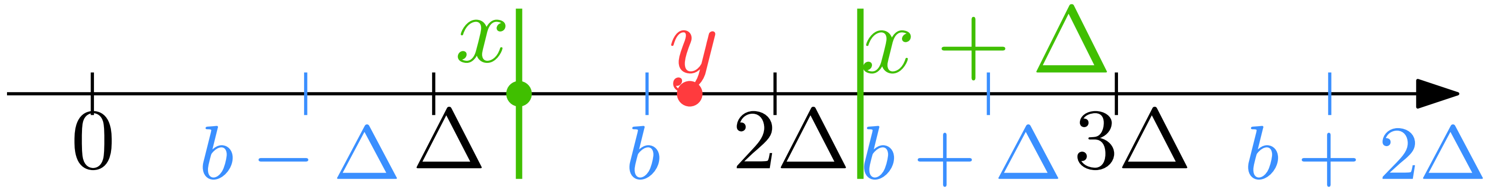

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. . . generalizes to grids and quadtrees . . .

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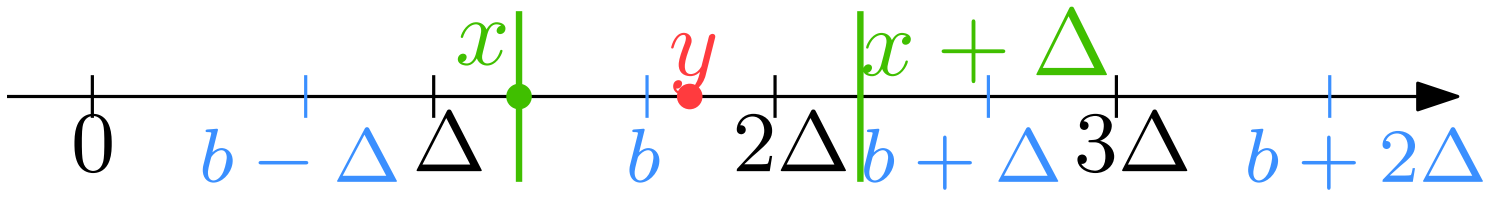

$$h_{b,\Delta}(x) = \lfloor \frac{x-b}{\Delta} \rfloor$$

Lemma: For $x, y \in \mathbb{R}$ holds $\mathbb{P}\left[h_{b,\Delta}(x) \neq h_{b,\Delta}(y)\right] = min\left(\frac{|x-y|}{\Delta}, 1\right)$

. . . generalizes to grids and quadtrees . . .

Lemma: Let s be a segment in the plane, The probability that s intersects the shifted grid of side length Δ is at most $\sqrt{2}\|s\|/\Delta$.

Shifted Grids

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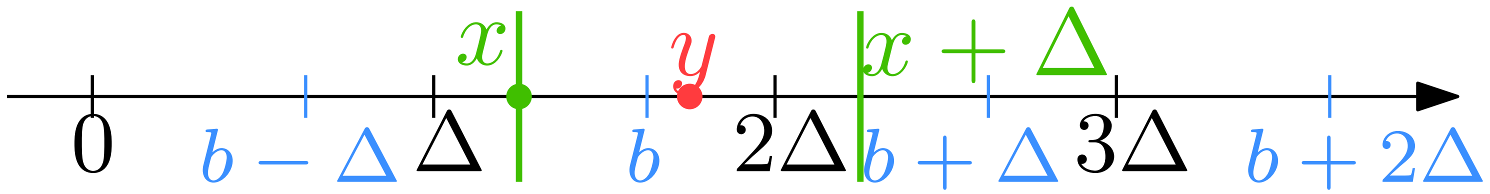

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Lemma: Let s be a segment in the plane, The probability that s intersects the shifted grid of side length Δ is at most $\sqrt{2}\|s\|/\Delta$.

Furthermore the expected number of intersection of s with vertical and horizonal lines of G_Δ is in the range $\left[\|s\|/\Delta, \sqrt{2}\|s\|/\Delta\right]$

Shifted Grids: Consequences

patching lemma + shifted grids $\rightarrow k$ portals per cell suffice

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add over H levels of quadtree

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Introduced error when:

- snapping to the grid
- bounding the number of intersections at k per side of each square
  uses: patching lemma (+ shifting)
- requiring the use of portals
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Introduced error when:

- snapping to the grid $1 + \frac{\varepsilon}{2}$
- bounding the number of intersections at k per side of each square $1 + \frac{8}{k-2}$
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Recall: $k = \frac{90}{\varepsilon}$ and $m \geq \frac{20H}{\varepsilon}$

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$$\left(1+\frac{\varepsilon}{2}\right)\left(1+\frac{8}{k-2}\right)\left(1+\frac{2H}{m+1}\right)||\pi_{\mathrm{OPT}}|| \leq \left(1+\frac{\varepsilon}{2}\right)\left(1+\frac{\varepsilon}{10}\right)^2||\pi_{\mathrm{OPT}}||$$

$$\leq (1+\varepsilon)||\pi_{\mathrm{OPT}}||$$

Summary

shifted quadtree with points snapped to grid

dynamic programming on quadtrees

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For a set P of n points in $\mathbb{R}^2$ and $\varepsilon > 0$, we can compute a tour π over P with expected length $(1+\varepsilon)||\pi_{\text{OPT}}||$ in time $(\varepsilon^{-1}\log n)^{\mathcal{O}(1/\varepsilon)}$