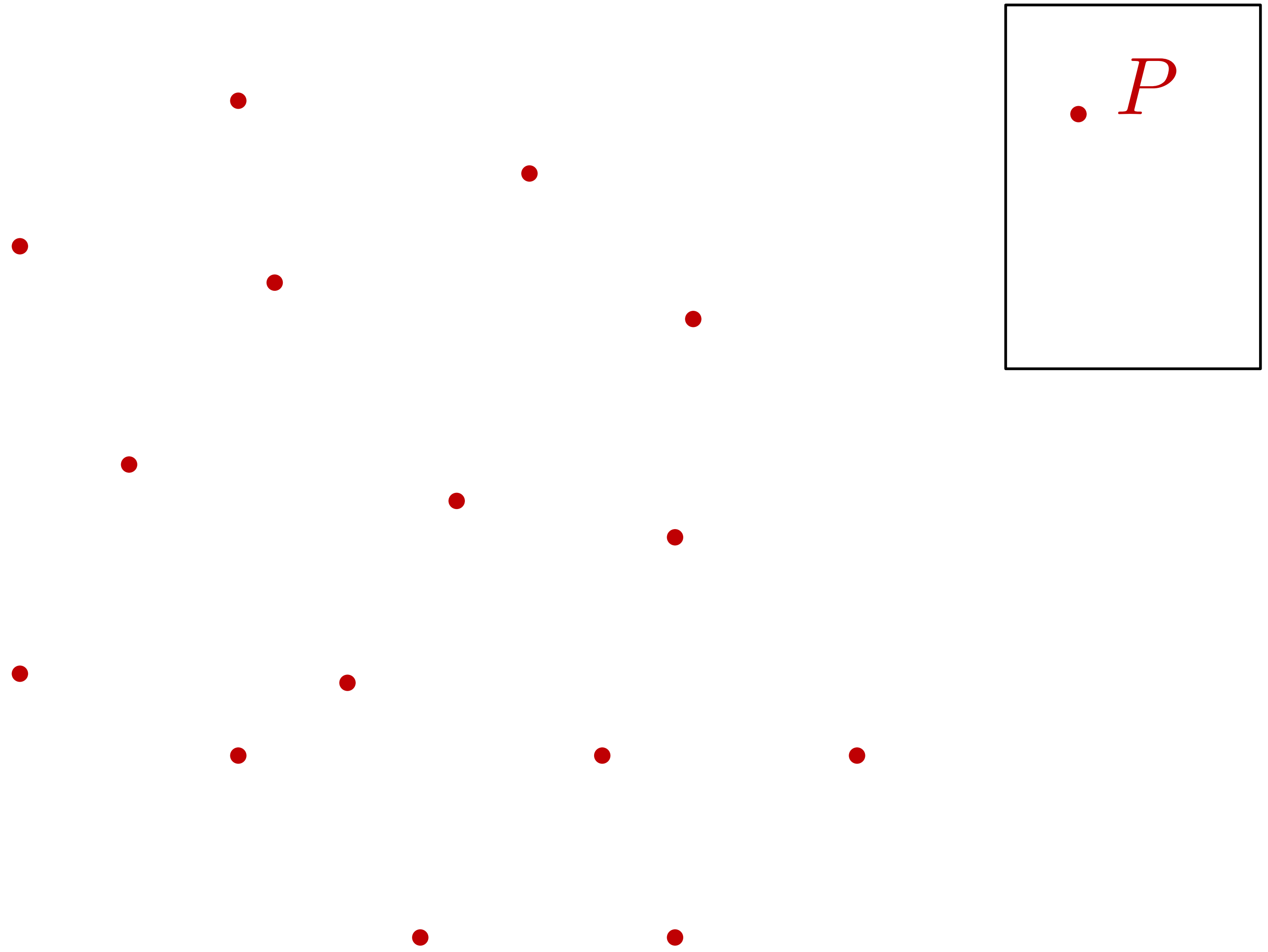
ε-sampling

range space
VC-dimension
ε-nets
ε-samples

Motivation: sampling for approximation

Given P,



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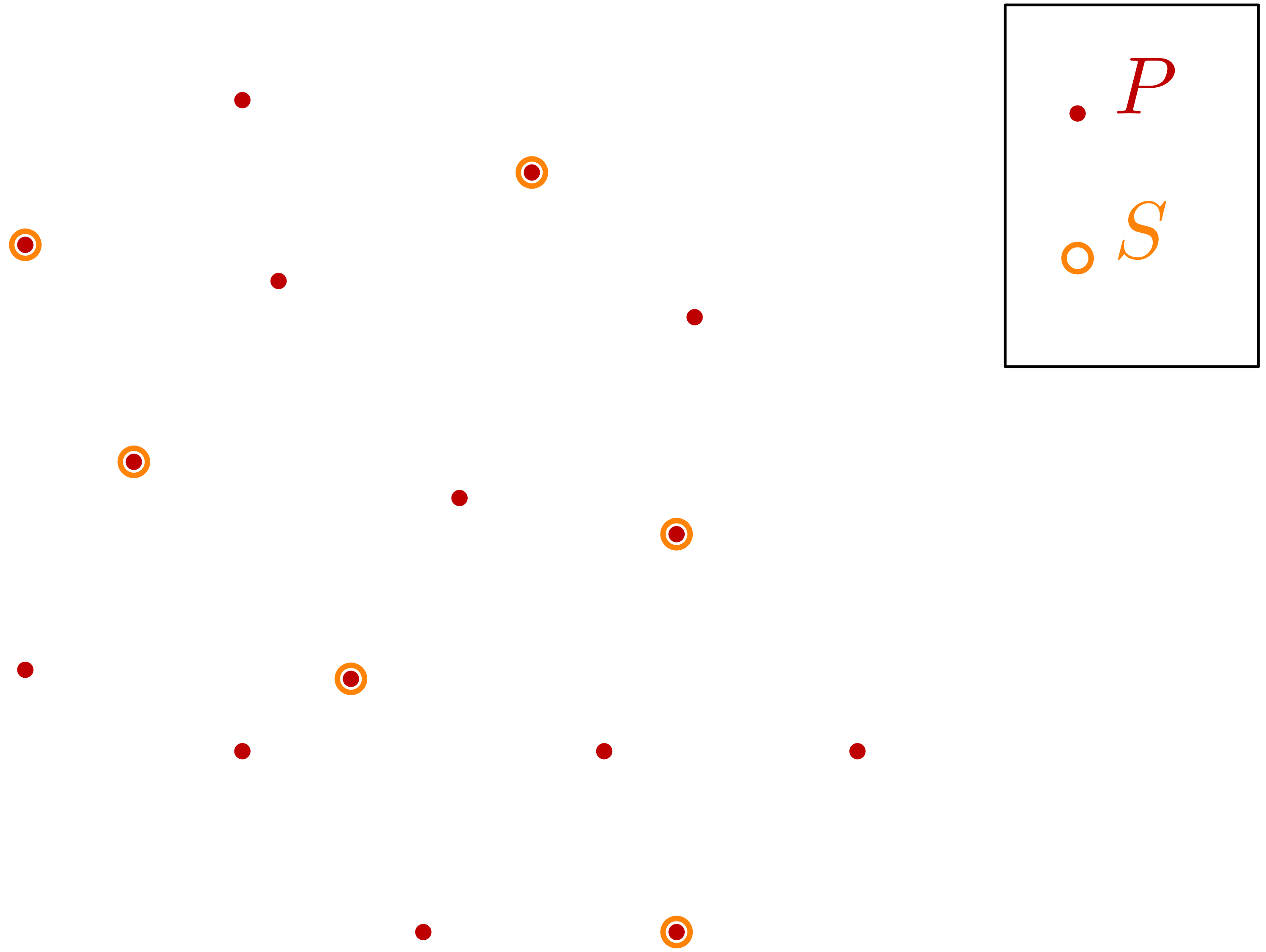
how many points do we need to sample ($S \subset P$), such that



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Given P,

how many points do we need to sample ($S \subset P$), such that

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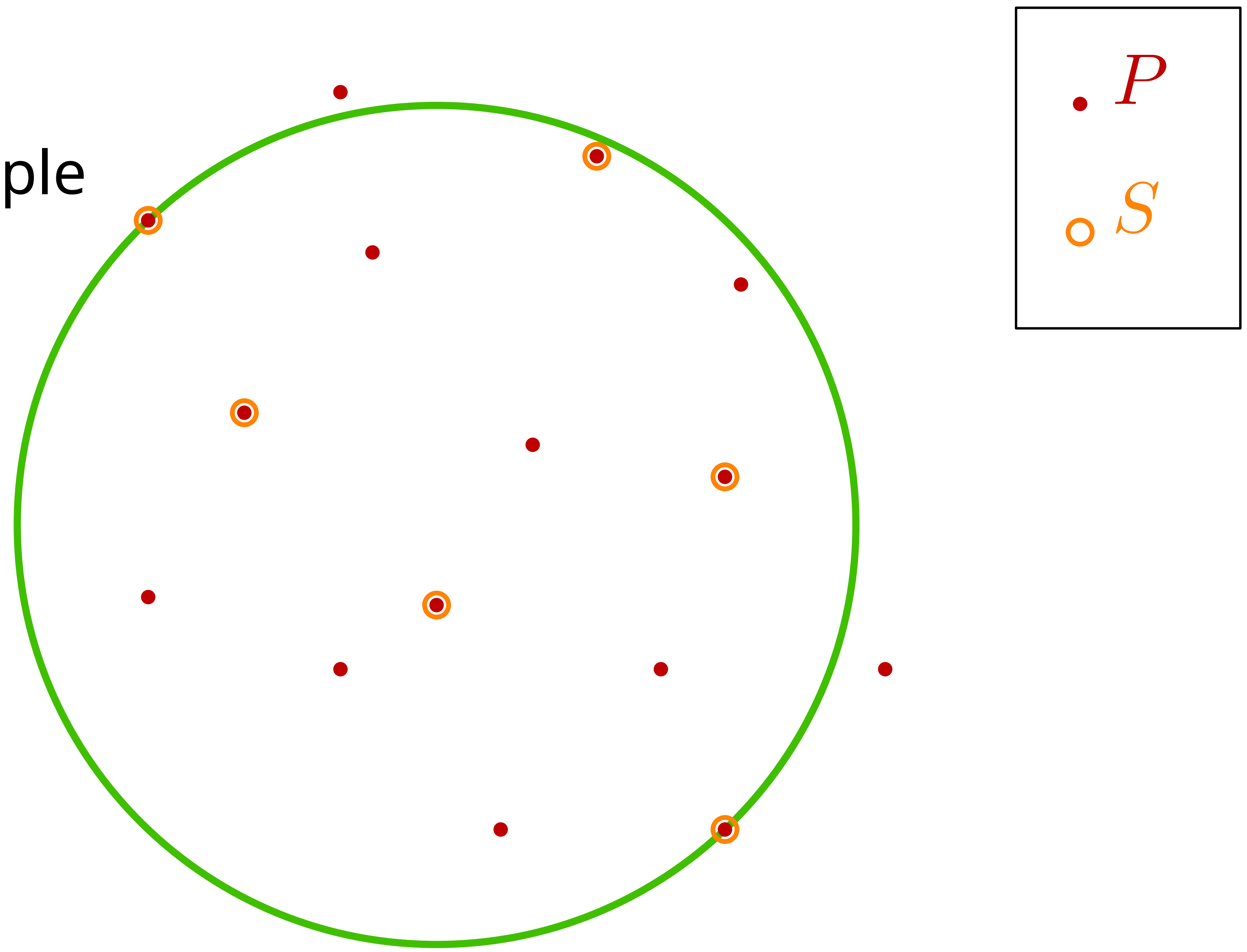

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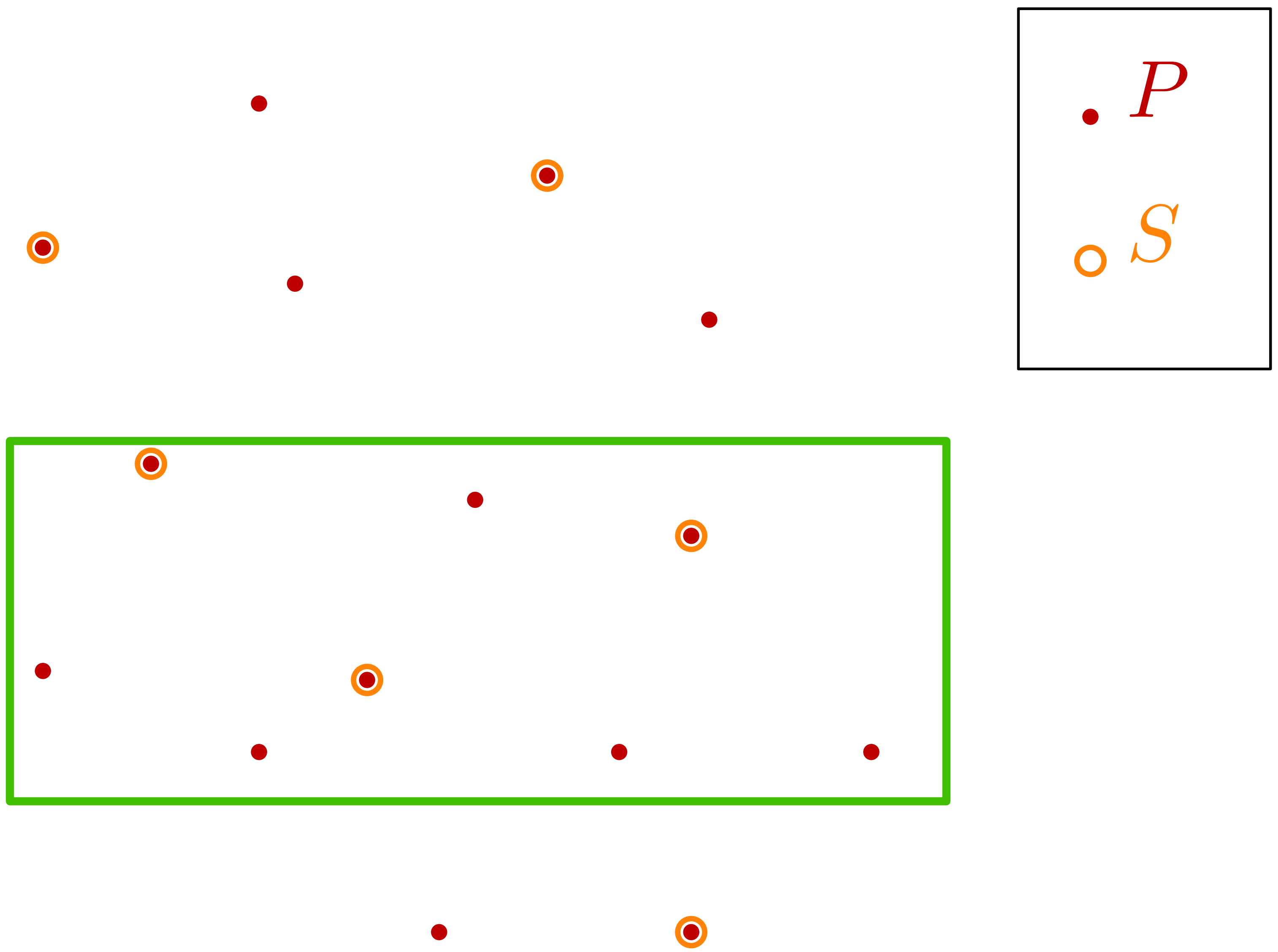
2. for any query rectangle r

we can estimate the number of points of P in r?



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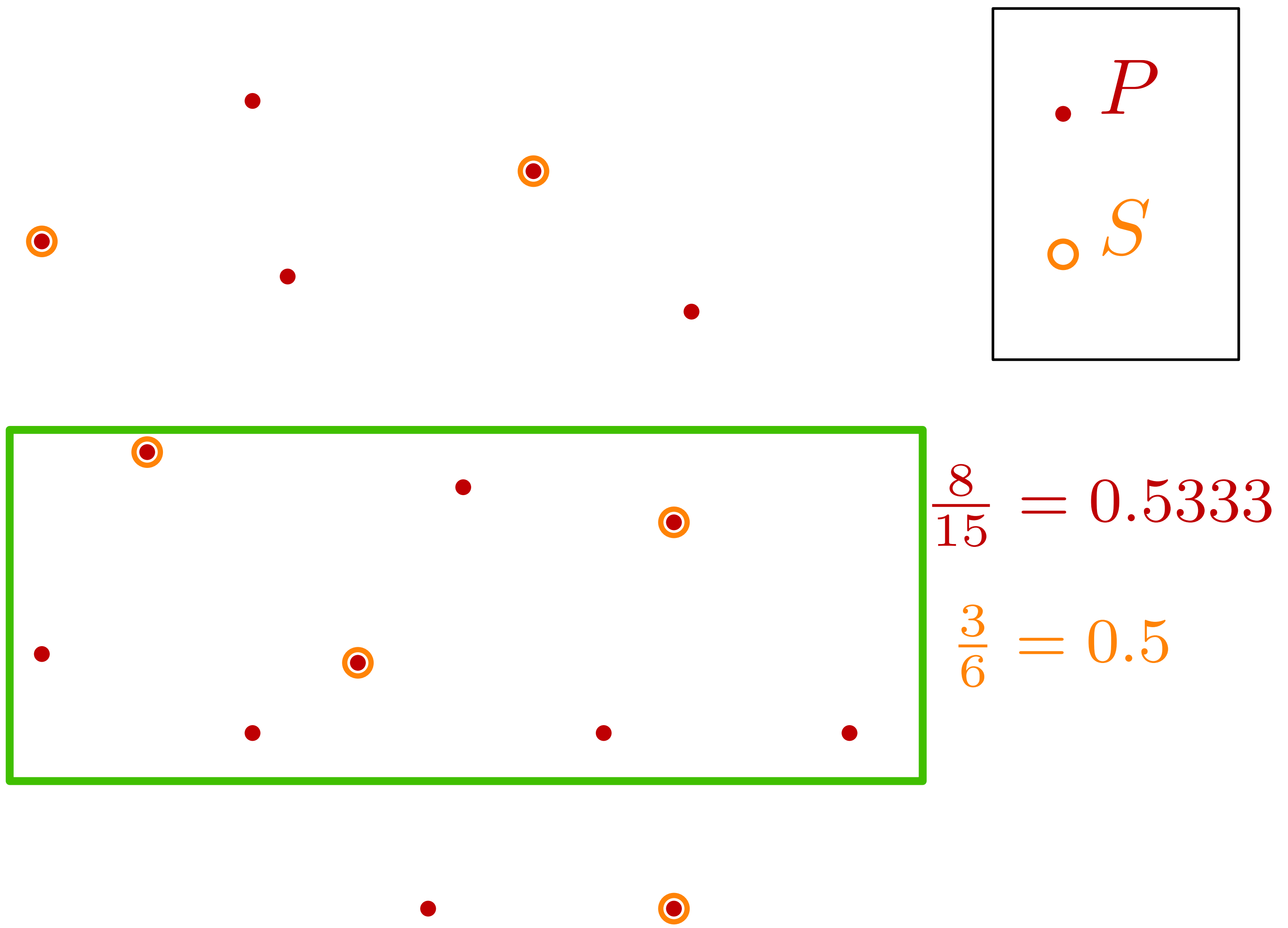
$$\left| \frac{|r \cap P|}{|P|} - \frac{|r \cap S|}{|S|} \right| \leq 0.25\,?$$



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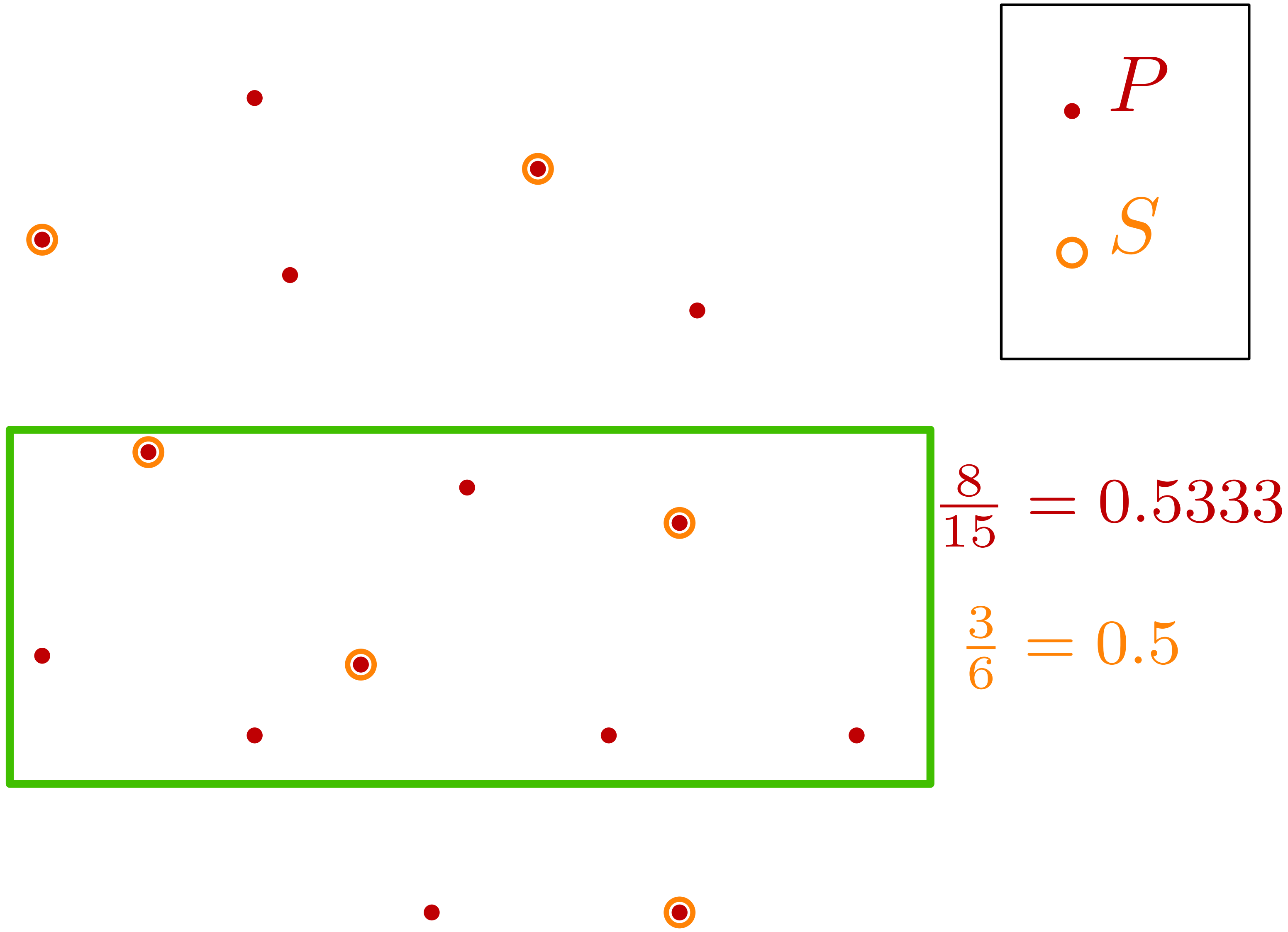
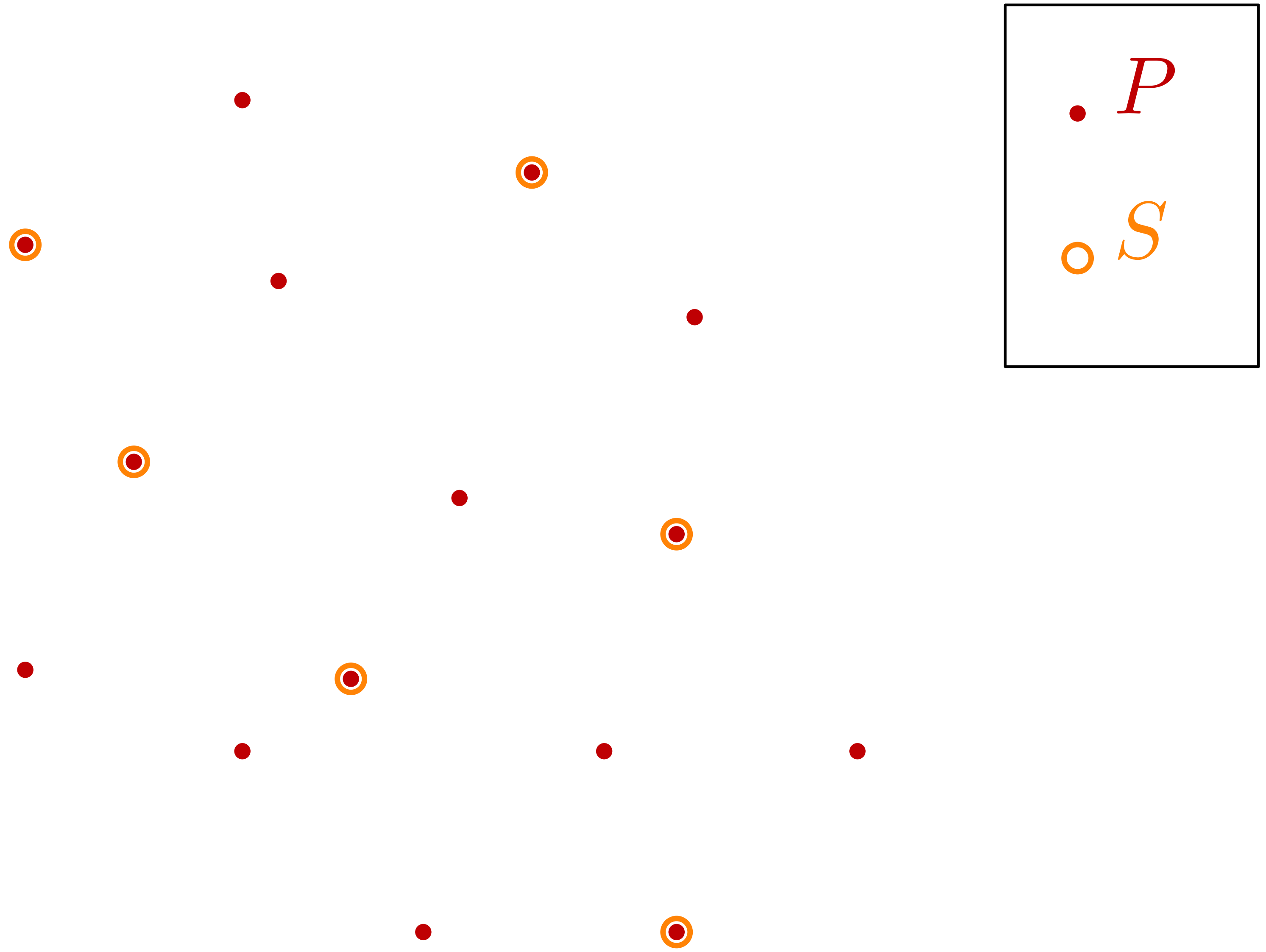


Ranges matter

$$\left|\frac{|r \cap P|}{|P|} - \frac{|r \cap S|}{|S|}\right| \leq 0.25$$ for all ranges r?



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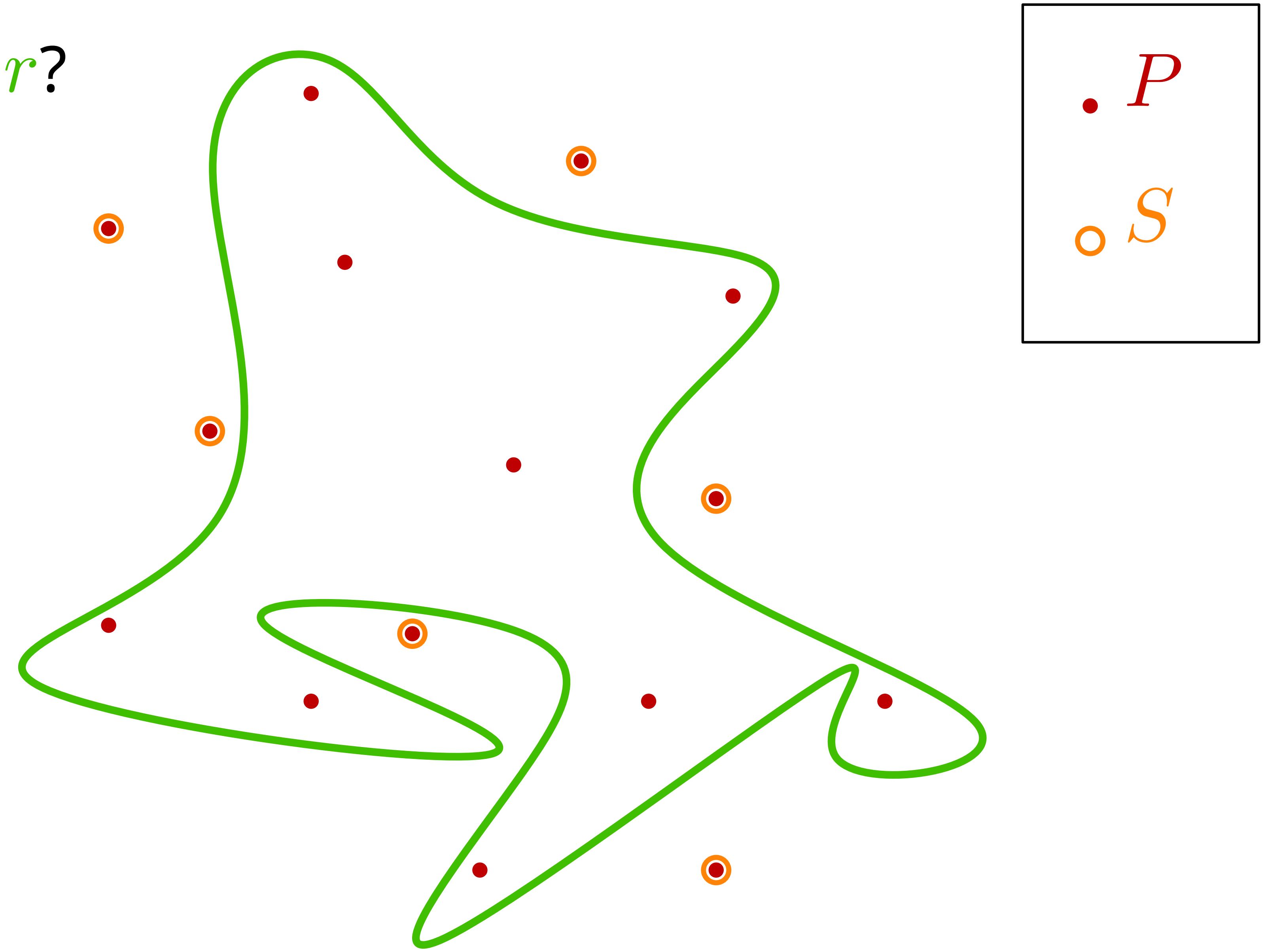

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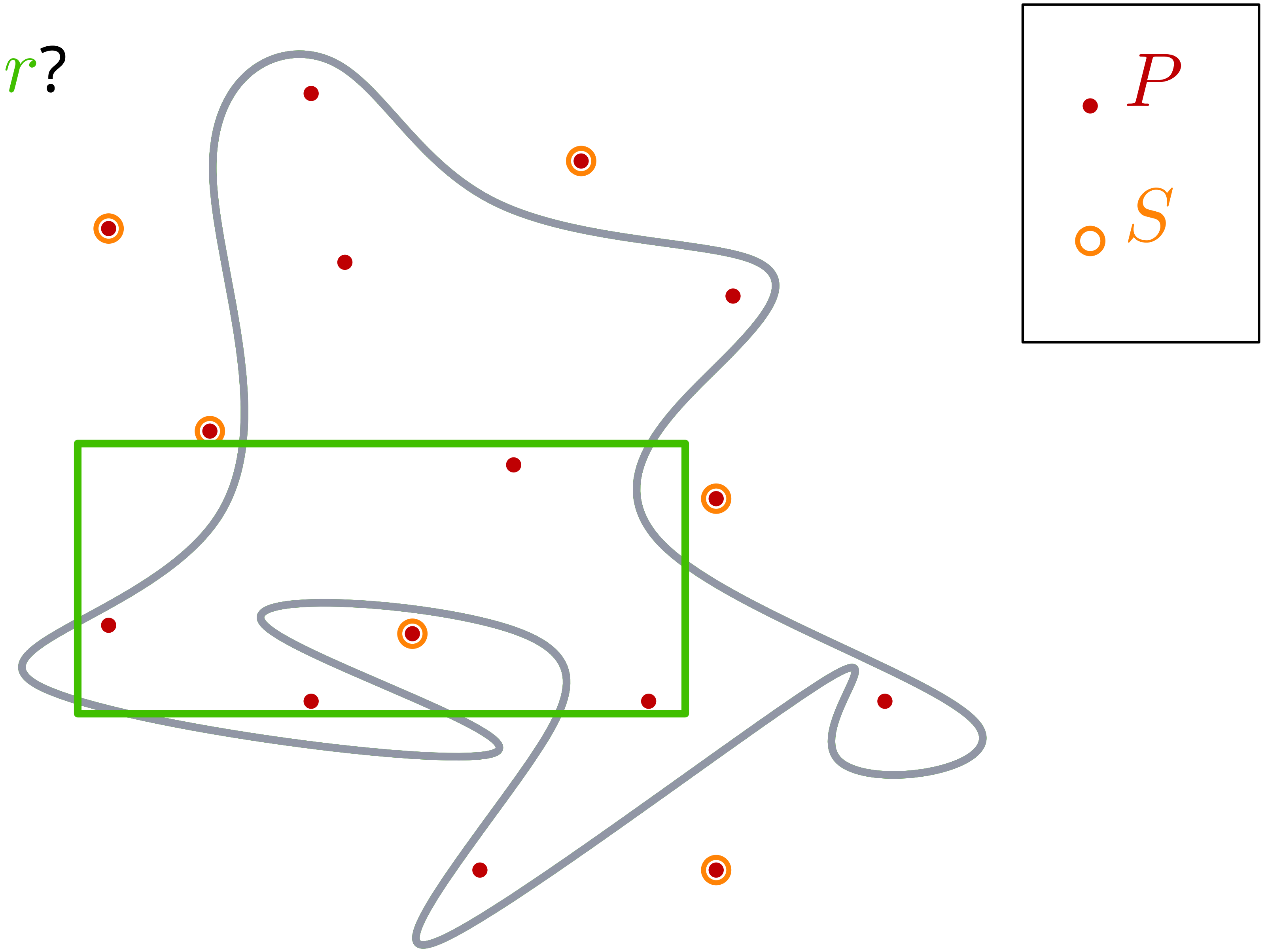

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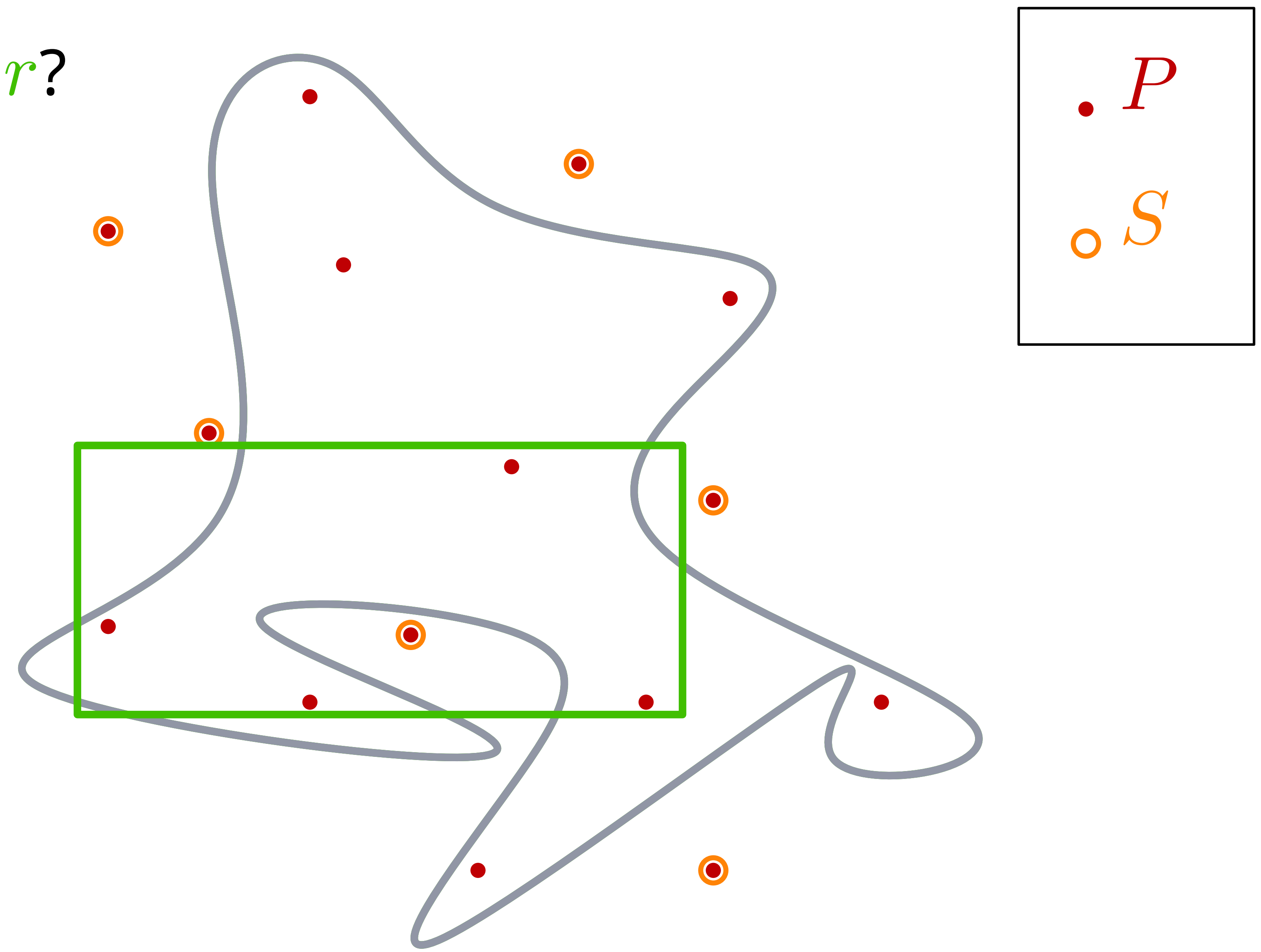
Ideas:

- for 5 points: range with 4 points will contain inner point



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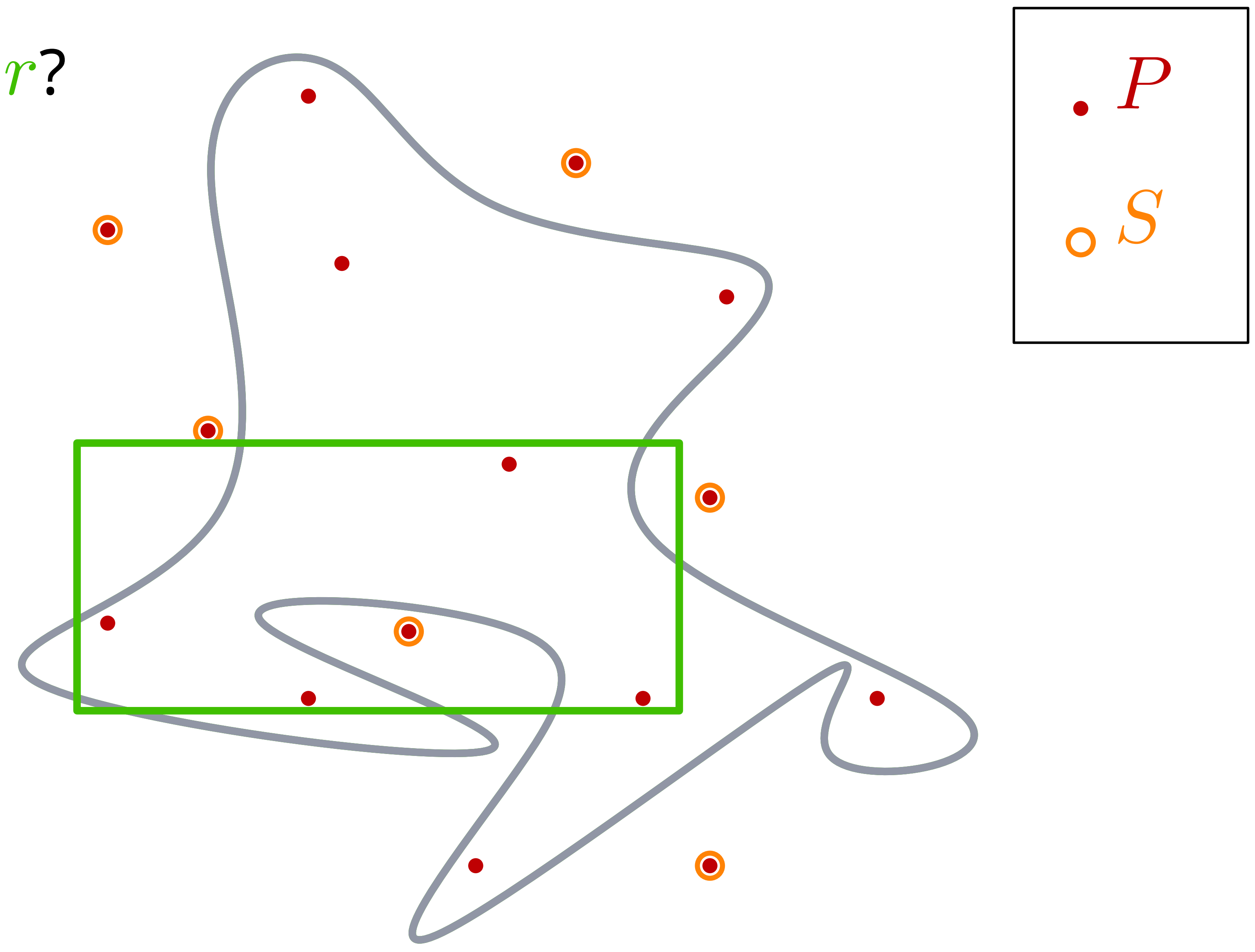
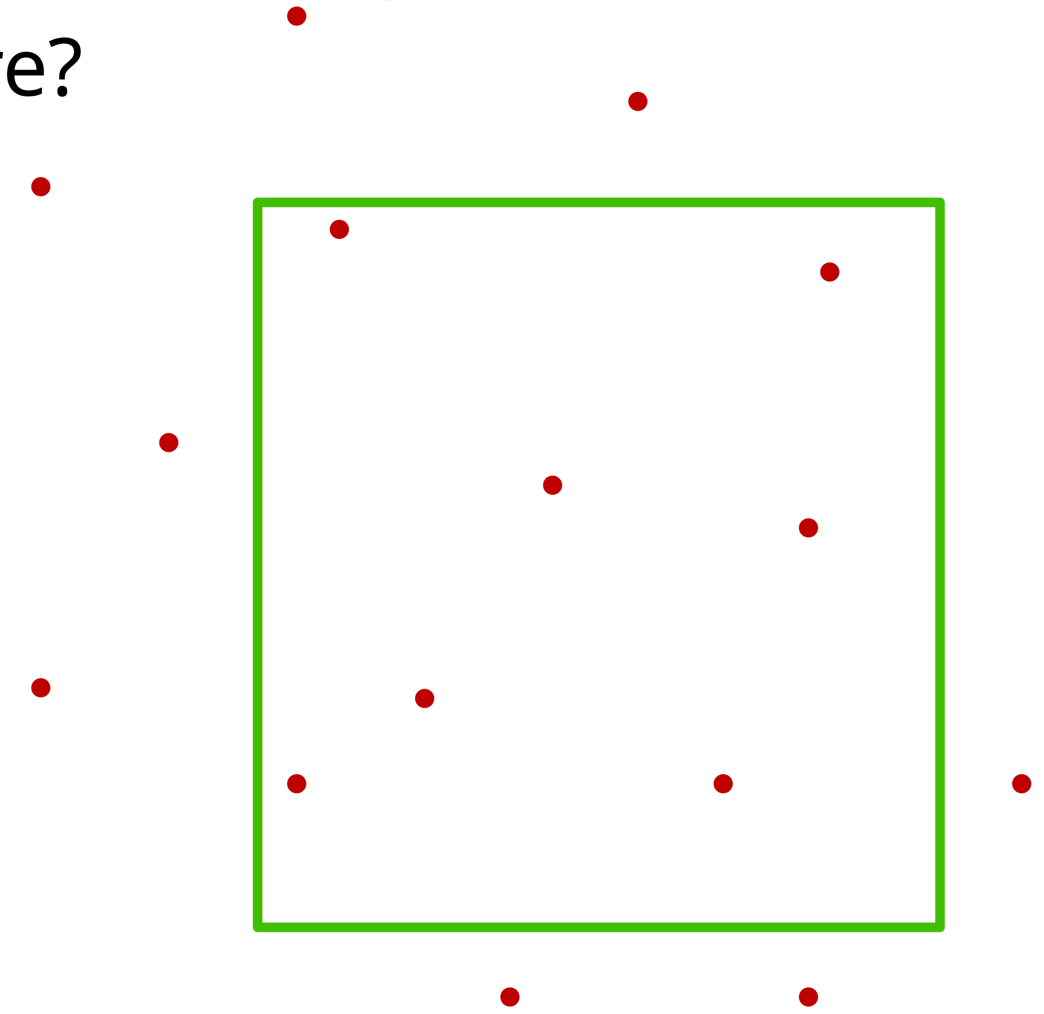
- 2^n subsets of P by general ranges but much fewer by rectangles



Quiz

Given point set P of size n and axis-aligned rectangles as ranges, how many sets $P \cap r$ are there?



A $O(n^2)$

B $O(n^3)$

C $O(n^4)$

(we ask for a tight bound)

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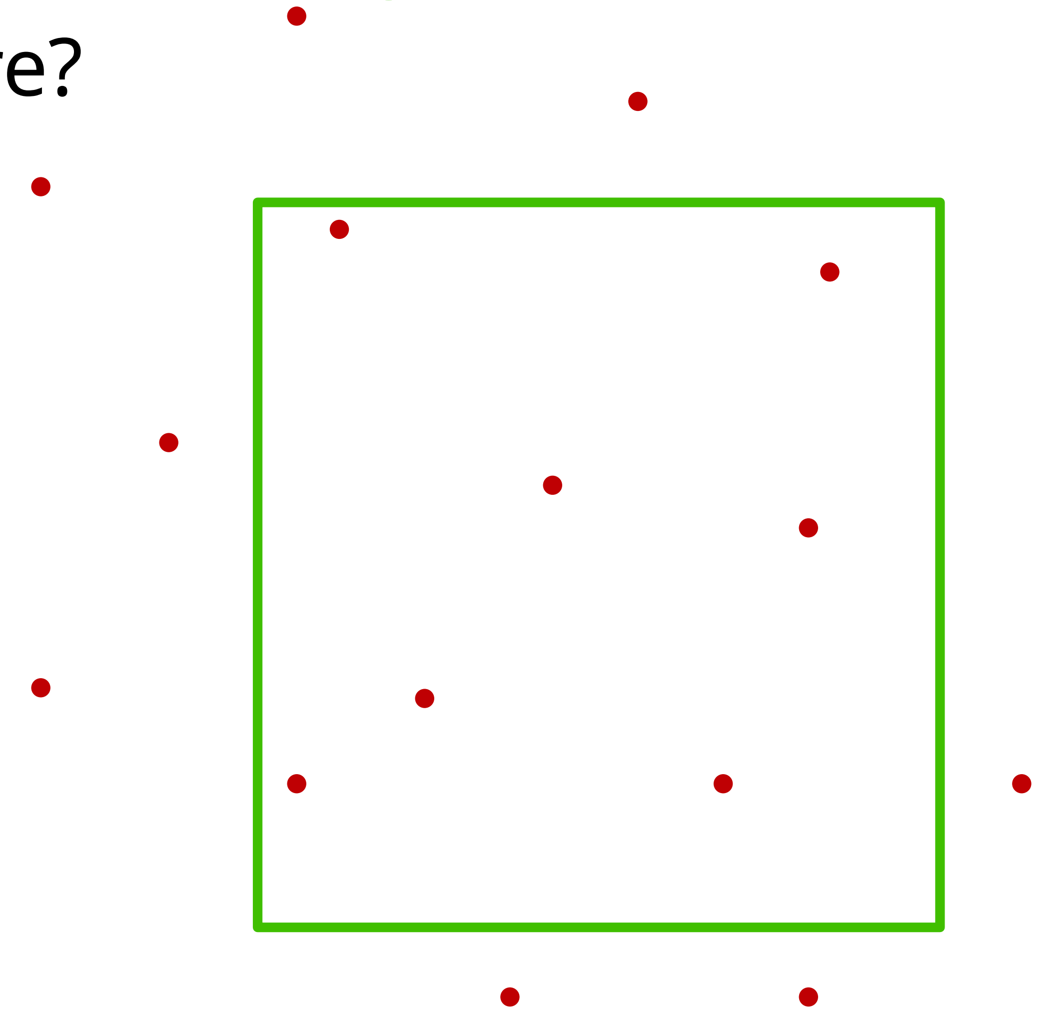

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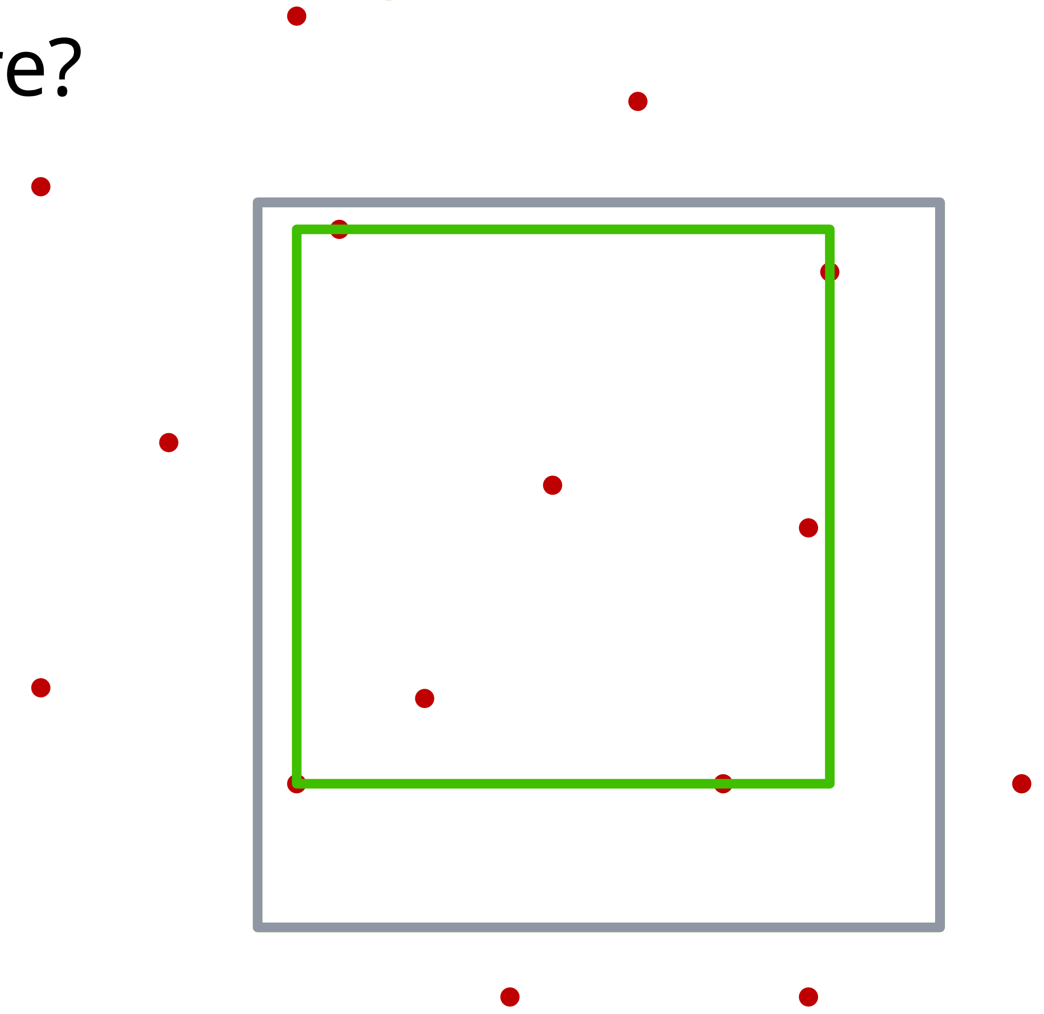

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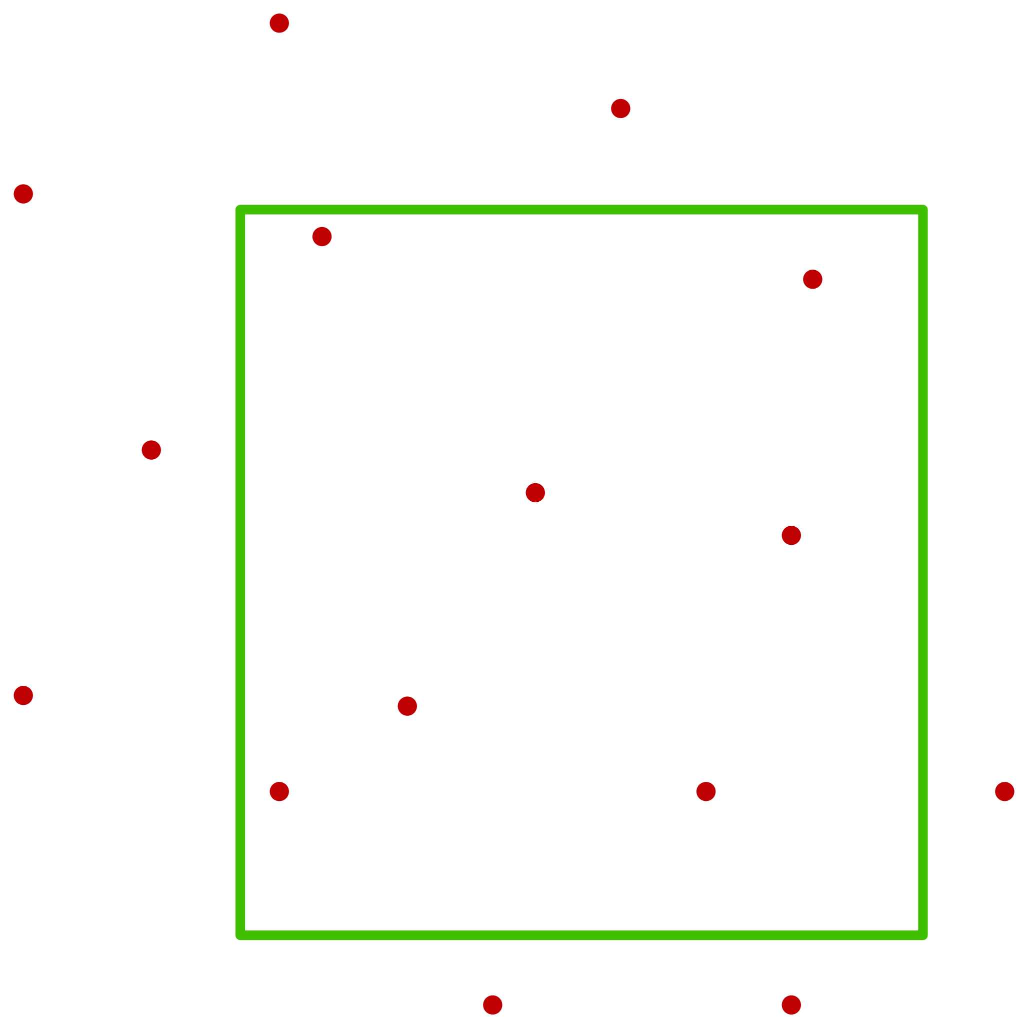
each minimal rectangle defined by left, top, right, bottom point

range spaces and VC-dimension

Range space

range space: pair $(X, \mathcal{R})$

- X is a set
- $\mathcal{R}$ is a subset of power set of X



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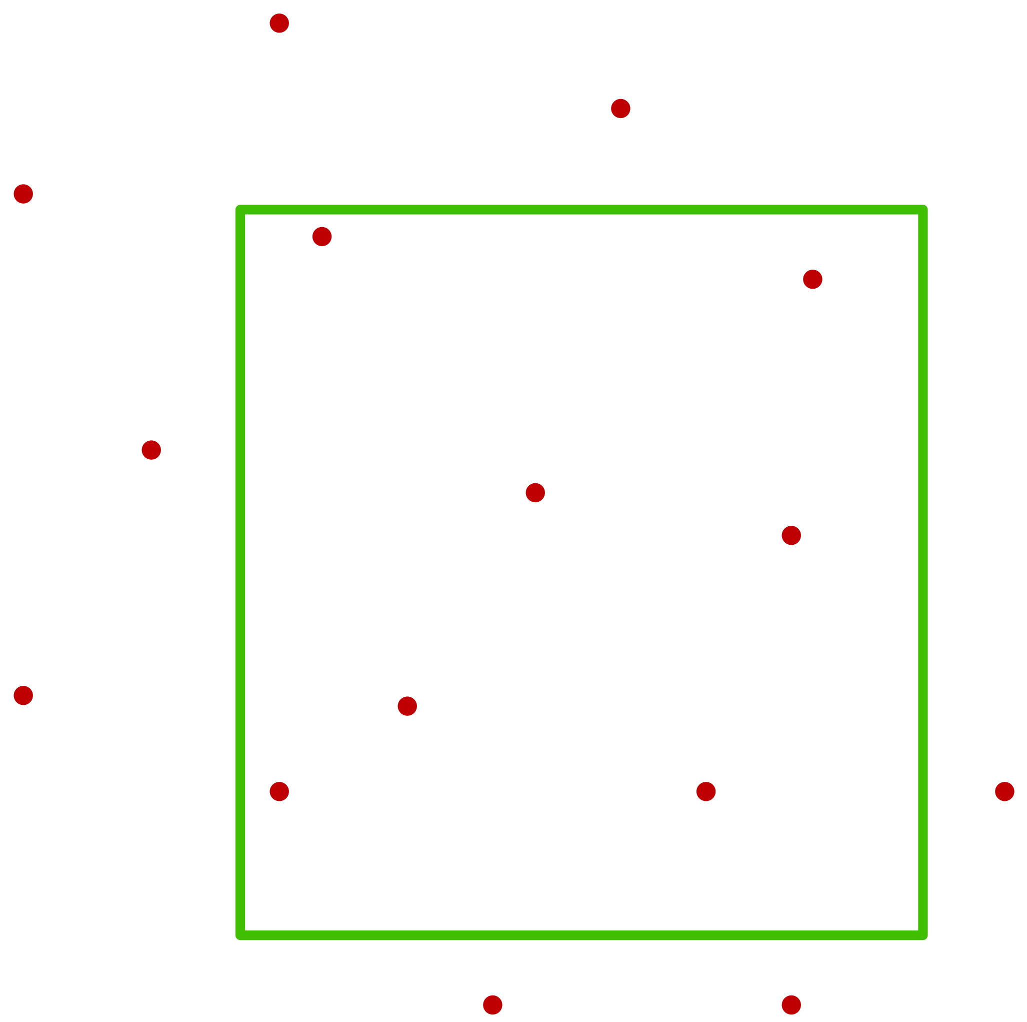

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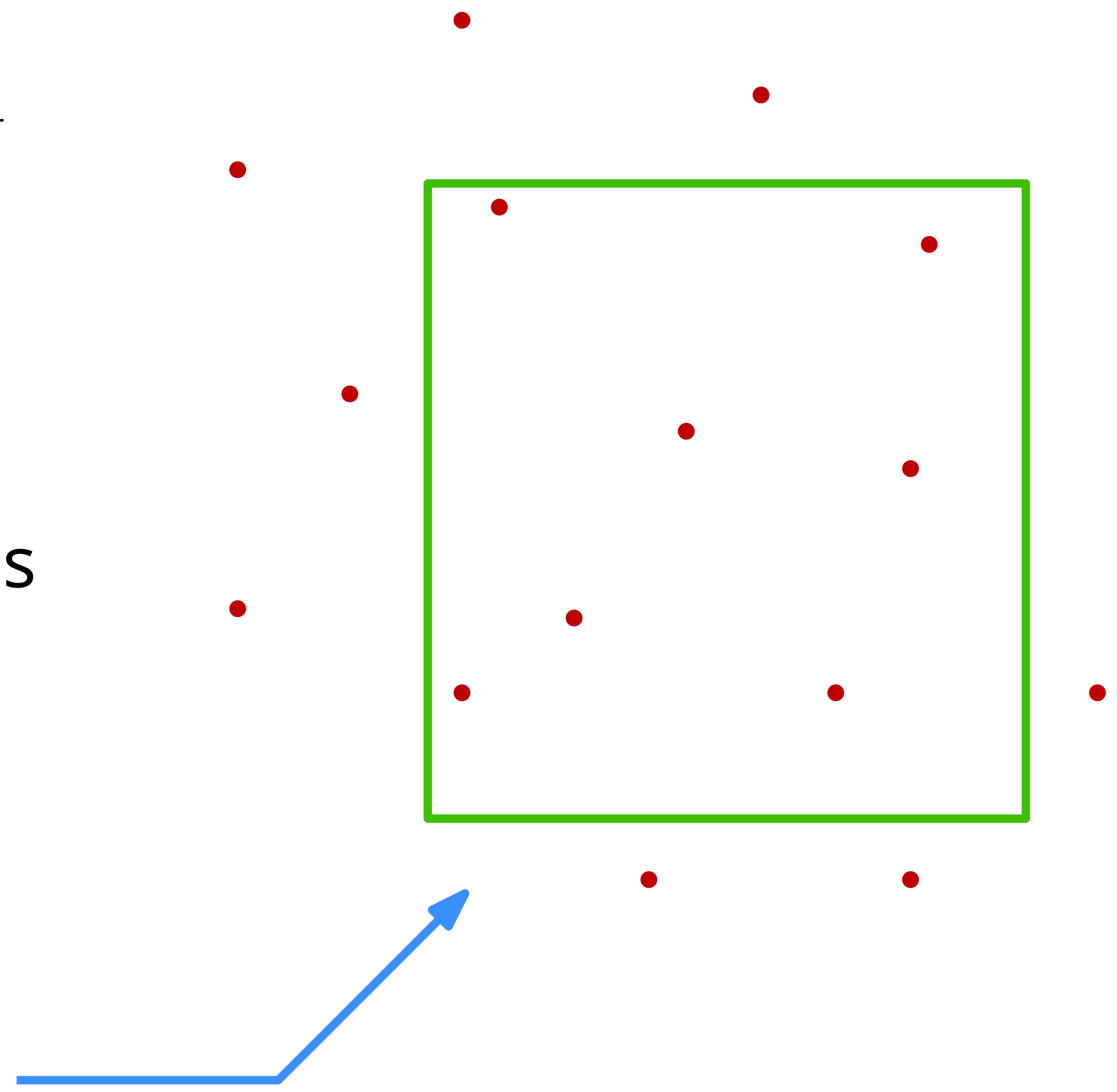

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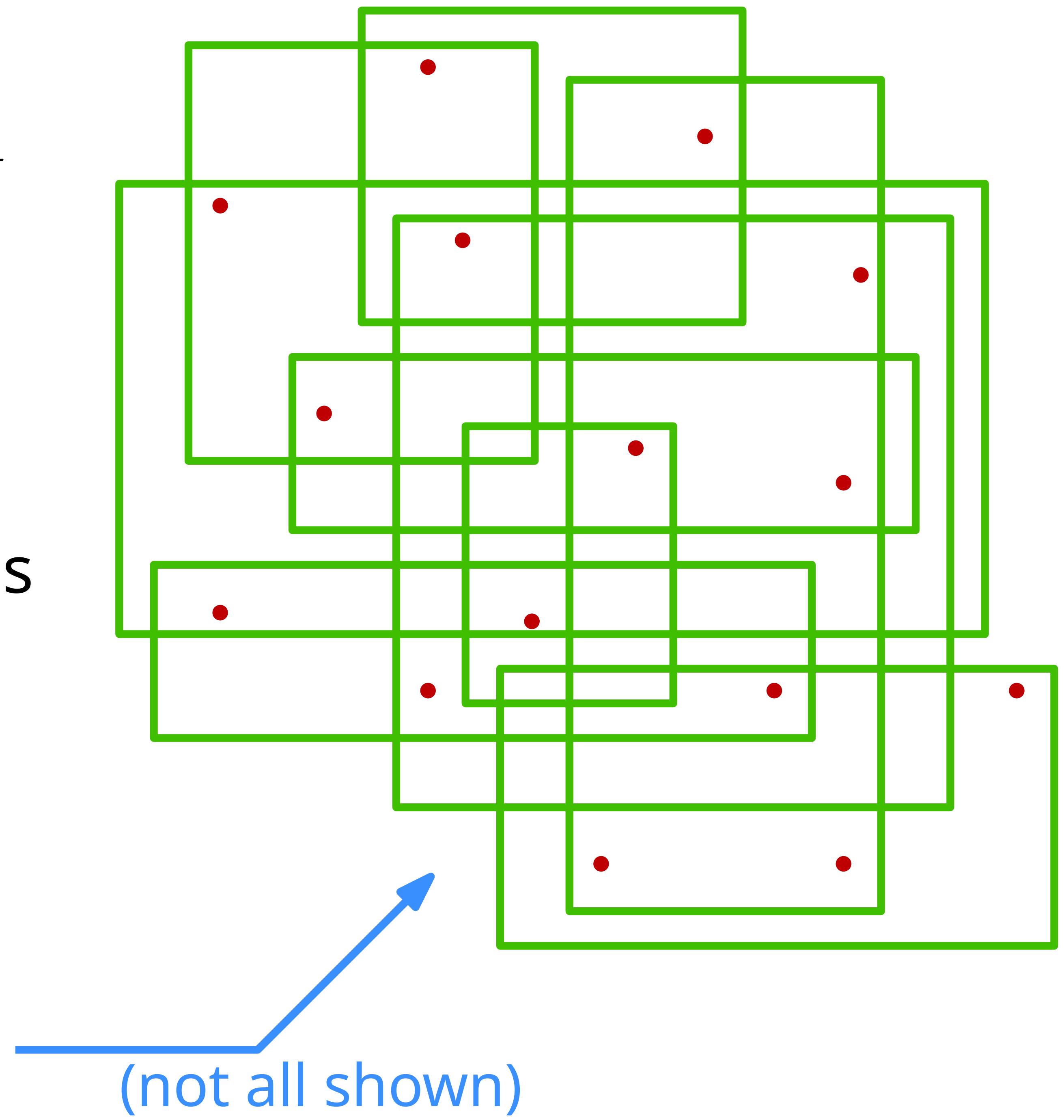


Examples of range spaces

$(\mathbb{R}, \mathcal{I})$, with $\mathcal{I} =$ set of closed intervals

$(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D} =$ set of disks

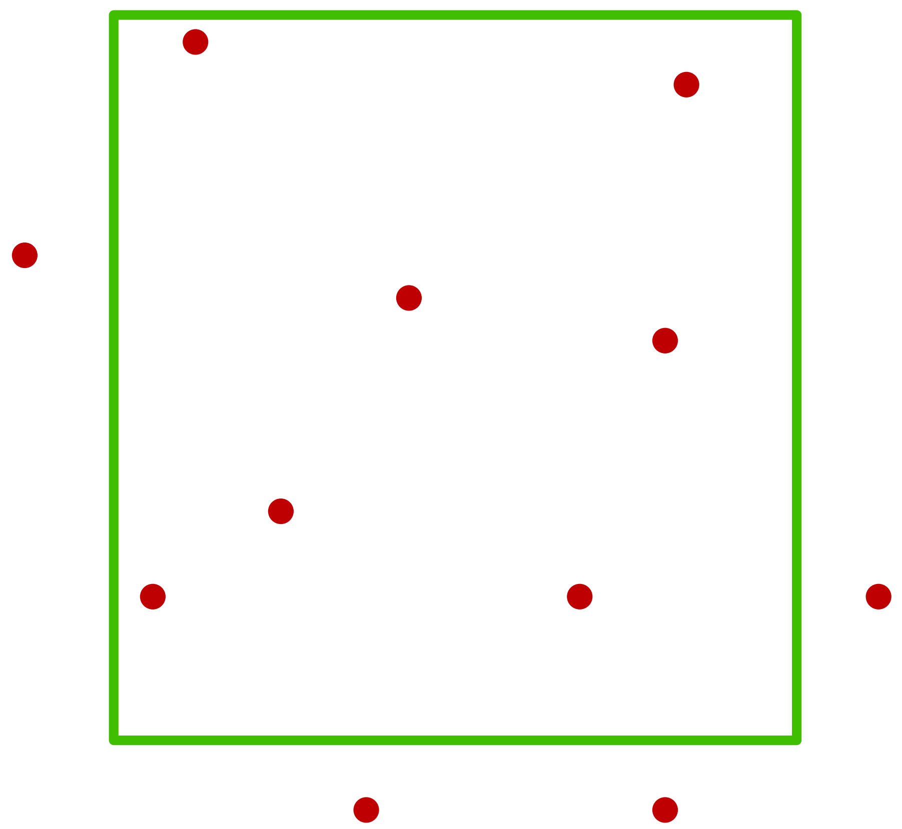
$(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR} =$ set of axis-aligned rectangles

$(\mathbb{R}^2, \mathcal{GR})$, with $\mathcal{GR} =$ set of arbitrary oriented rectangles

$(\mathbb{R}^2, \mathcal{C})$, with $\mathcal{C} =$ set of closed convex sets

VC-dimension

example: $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles



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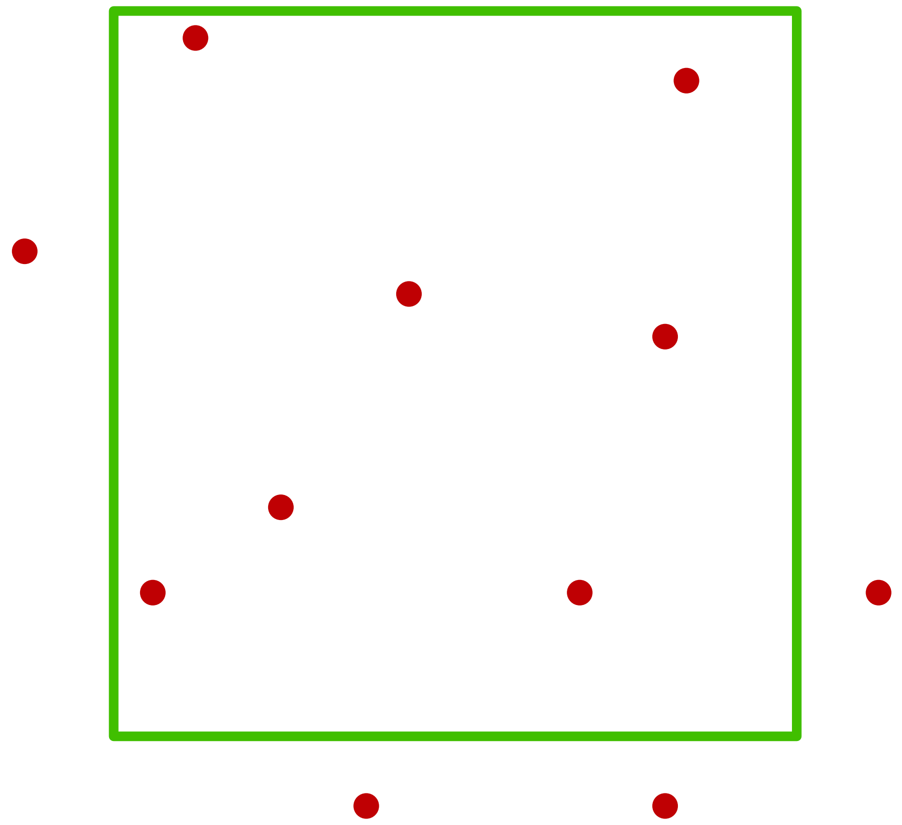

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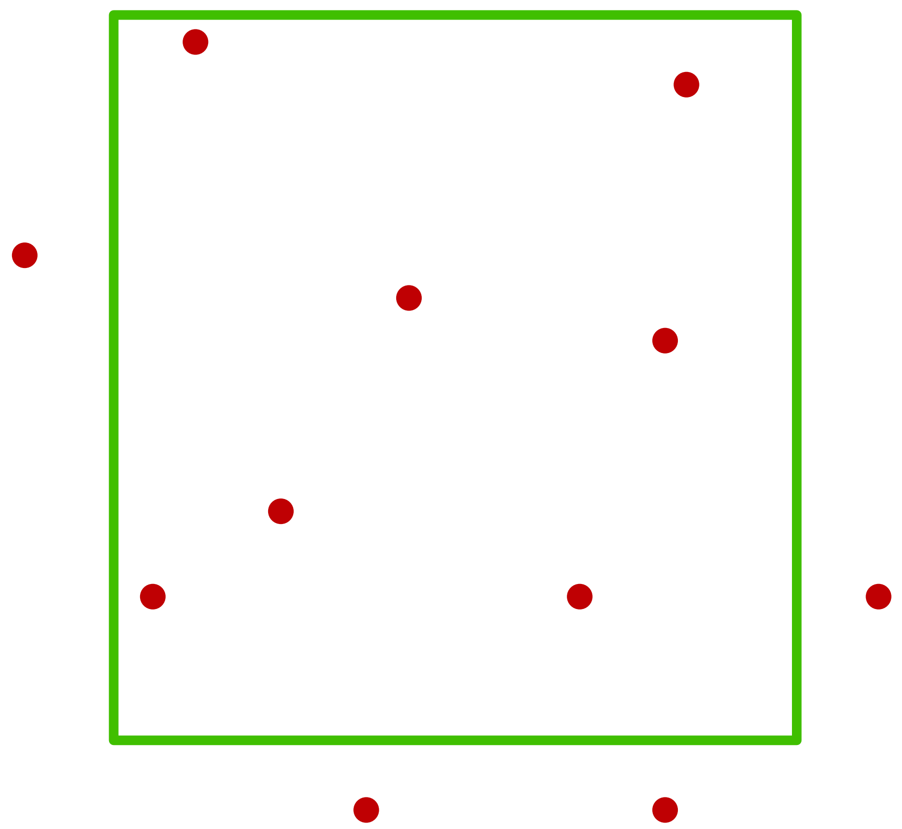

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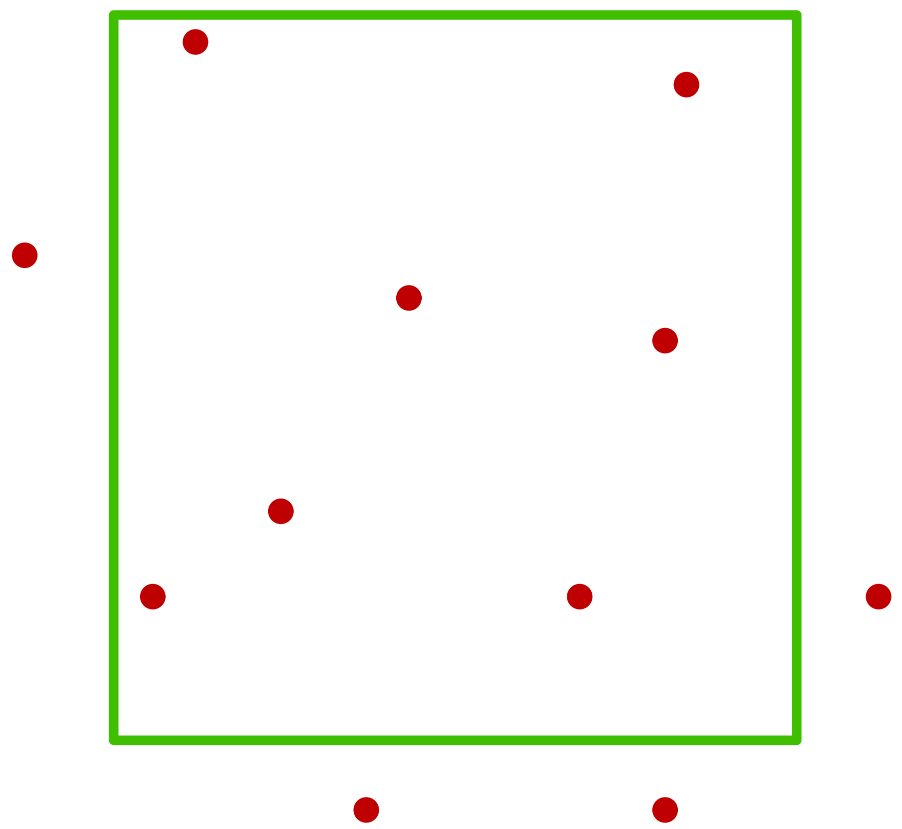

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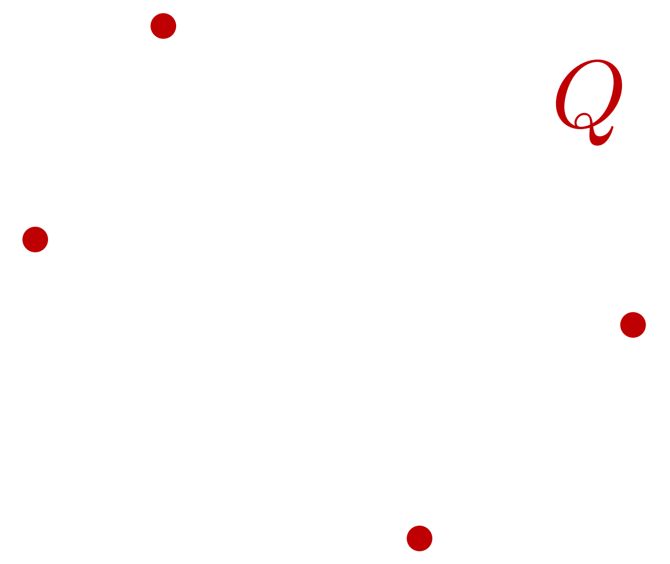

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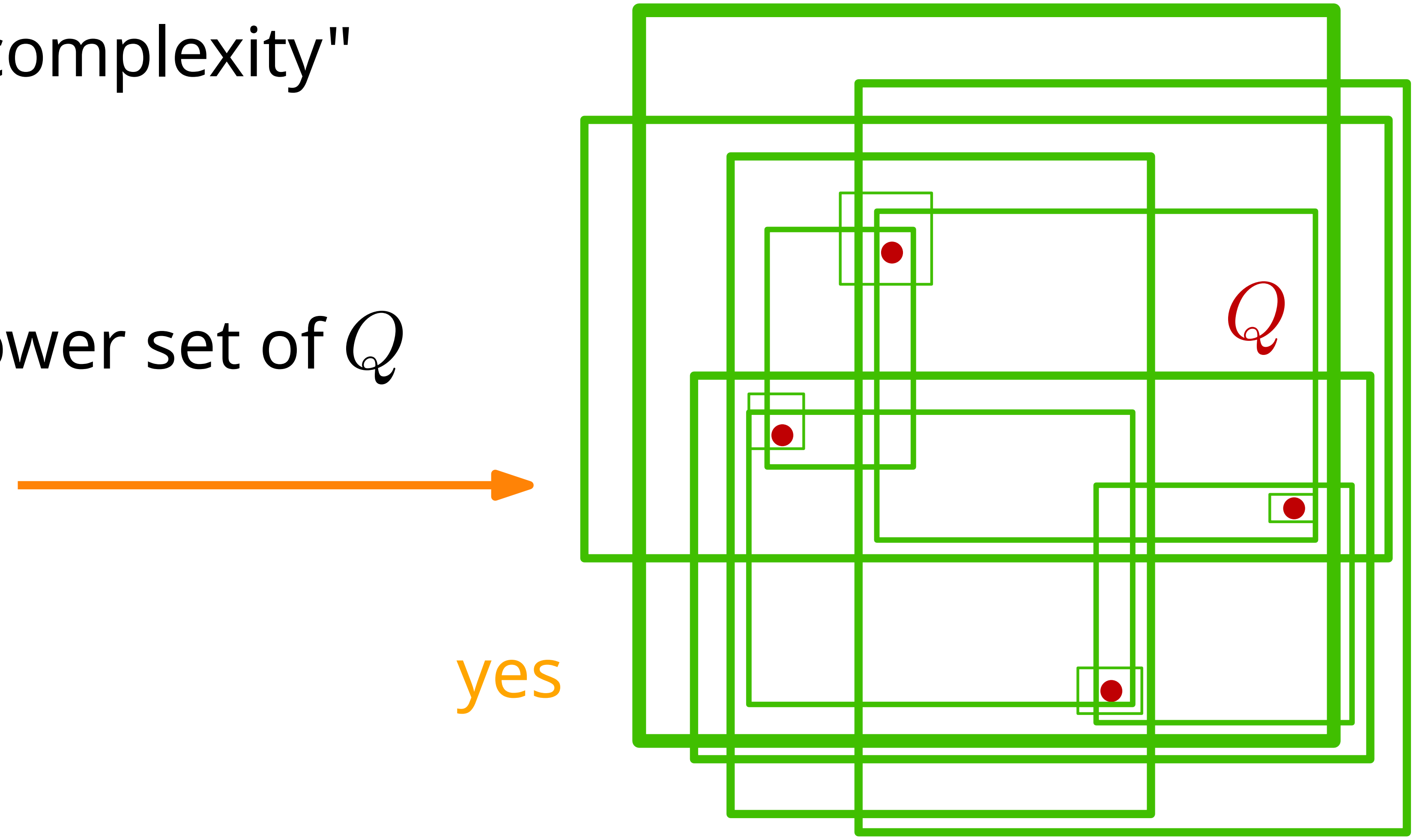

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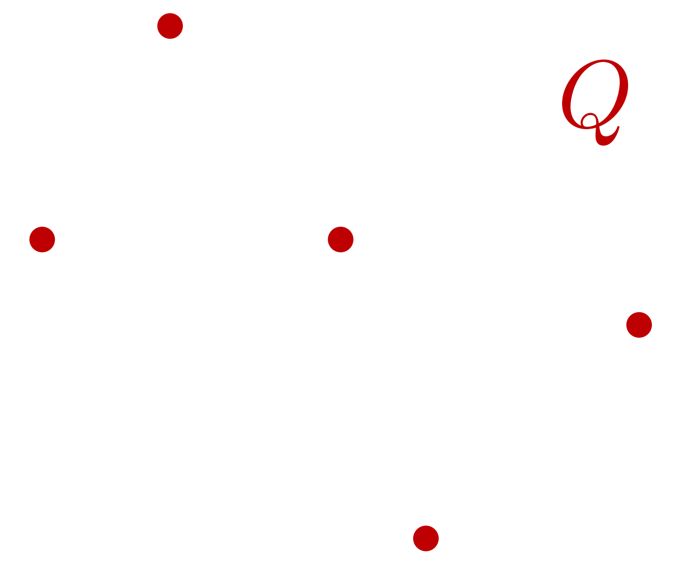

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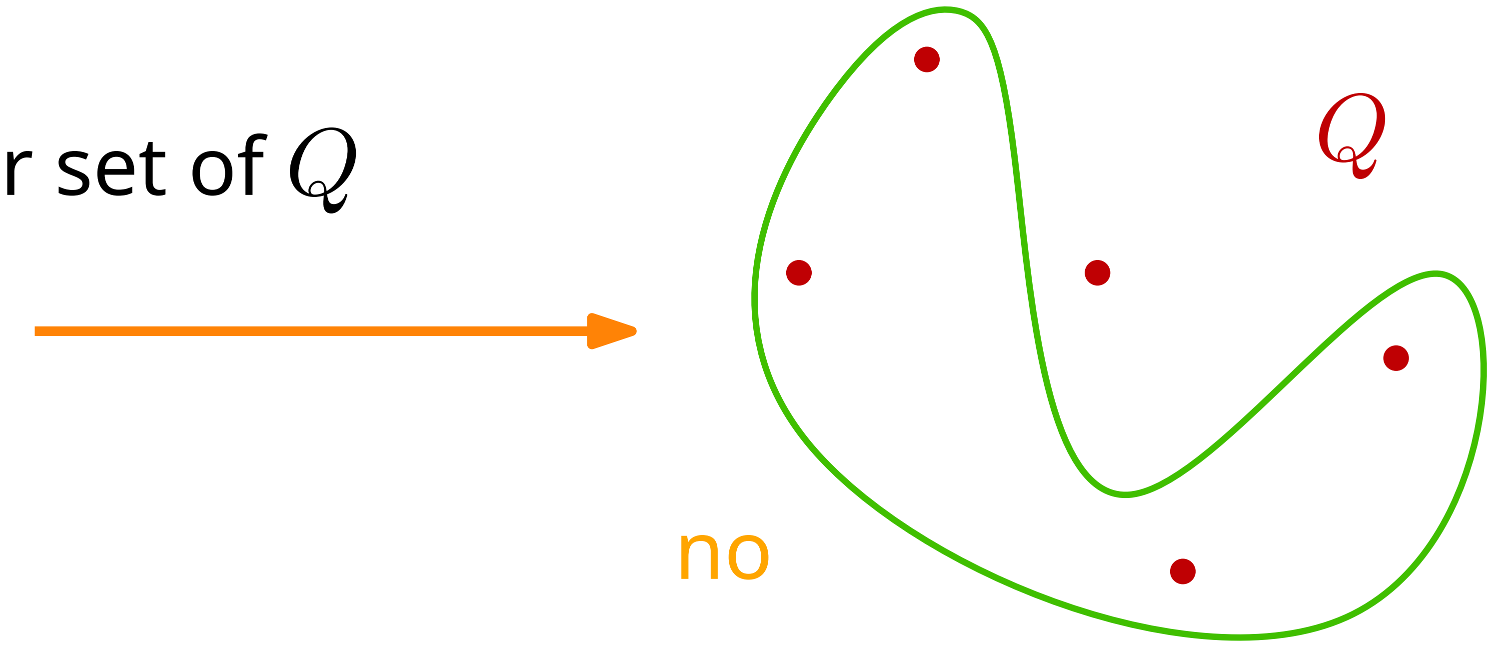

no

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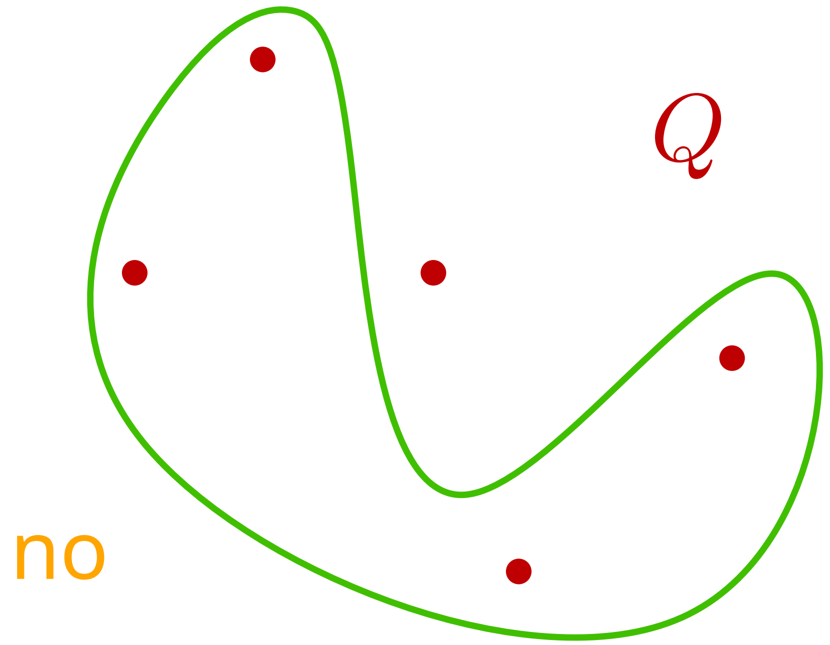

VC-dimension of a range space:
maximum size of a shattered subset of X

Example $(\mathbb{R}, \mathcal{I})$

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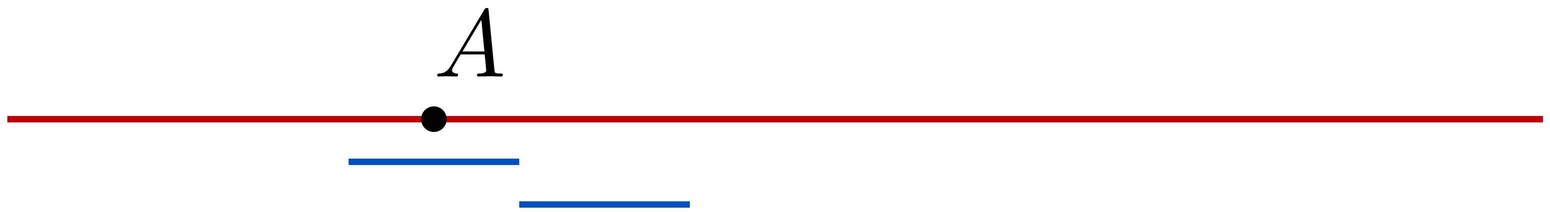

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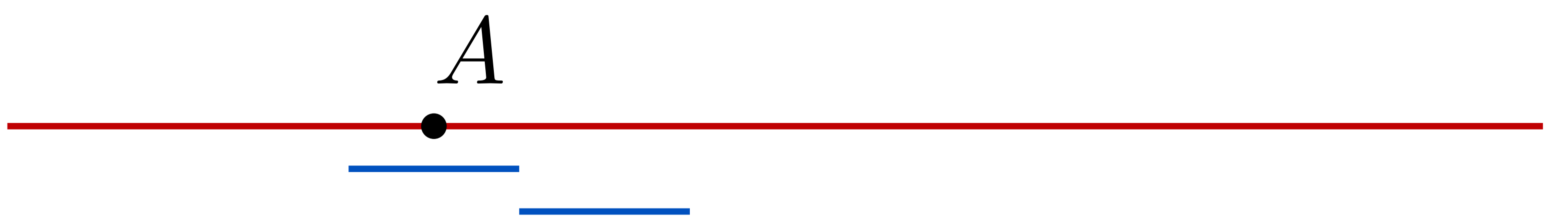

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$$\left|\mathcal{R}_{|P}\right| = 2 = 2^{|P|}$$

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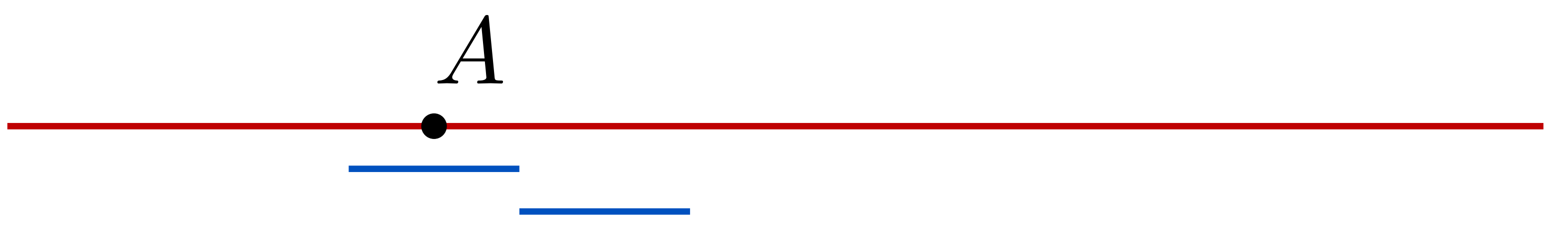

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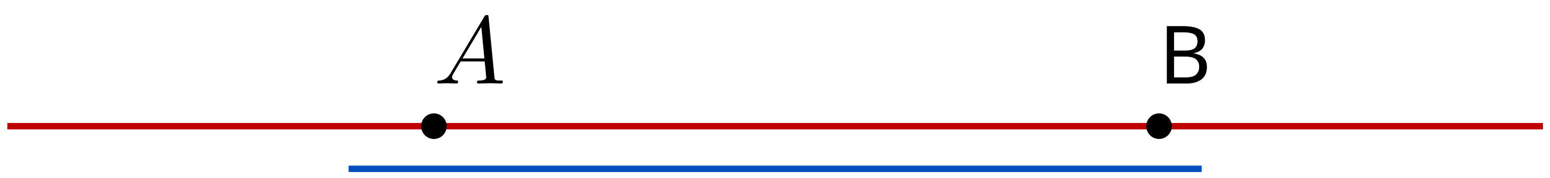

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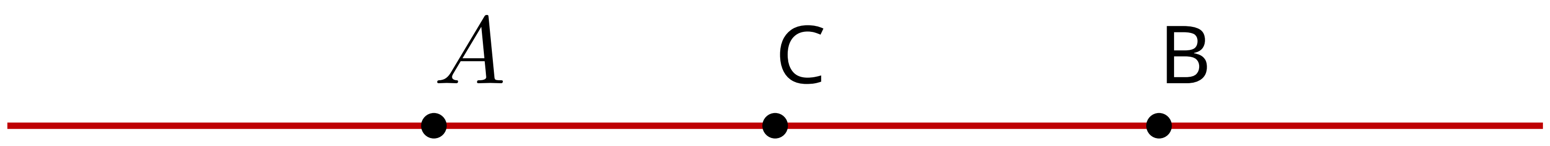

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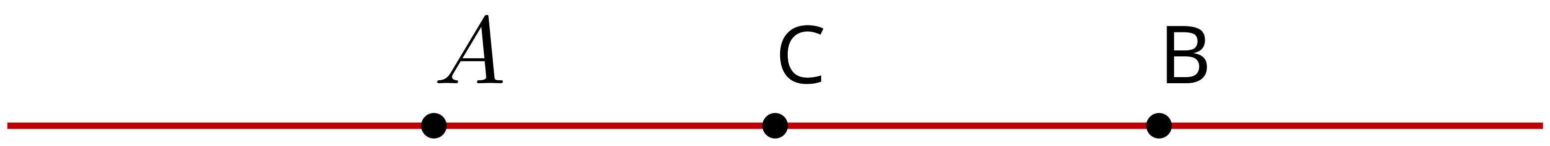

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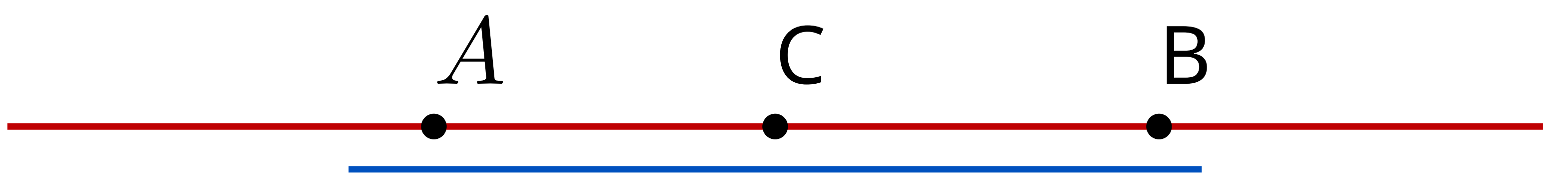

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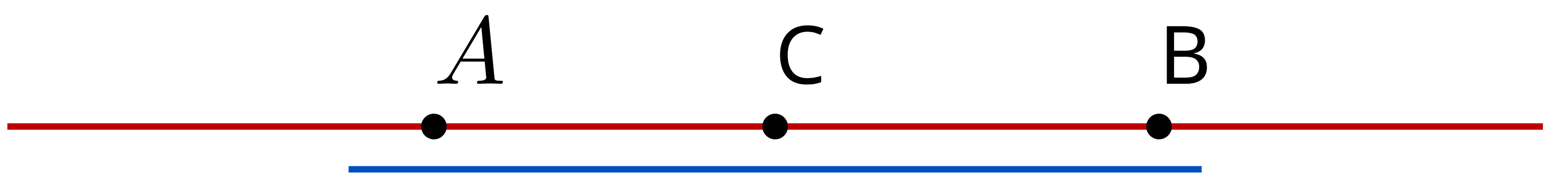

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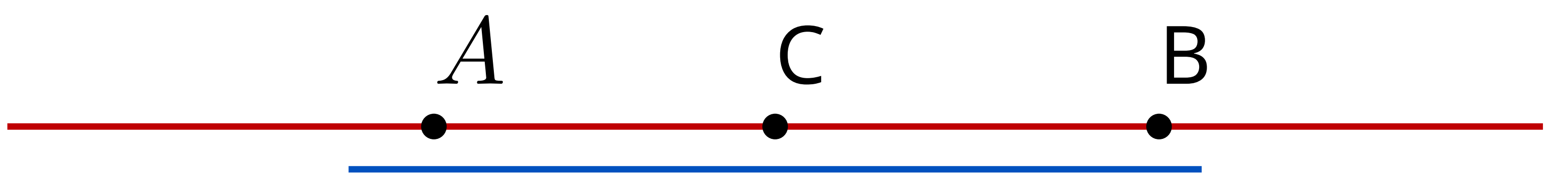

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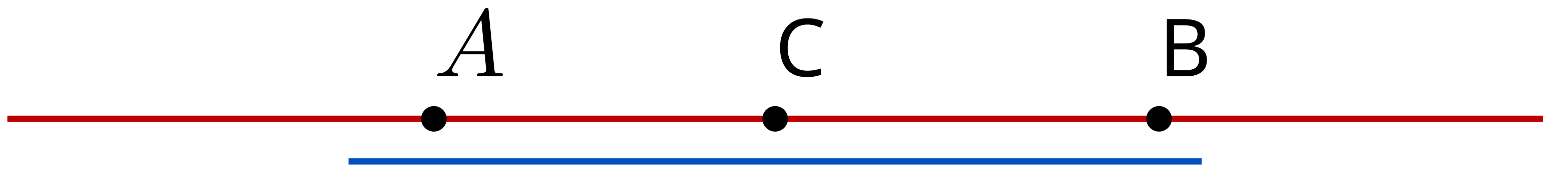

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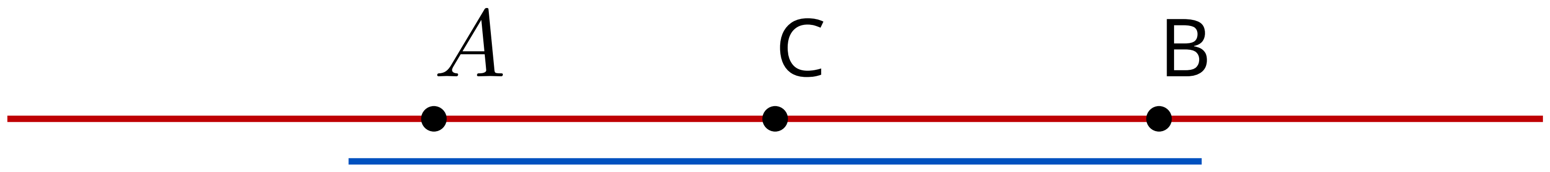

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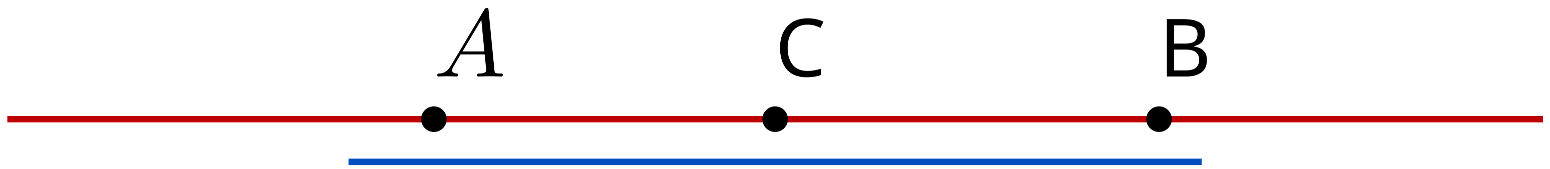

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VC-dimension $= 2$

Quiz

range space $(\mathbb{R}, \mathcal{I}_{\rightarrow})$ with $\mathcal{I}_{\rightarrow} = \{[a, \infty) | a \in \mathbb{R}\}$

What is the VC-dimension of this space?

A 1

B 2

C 3

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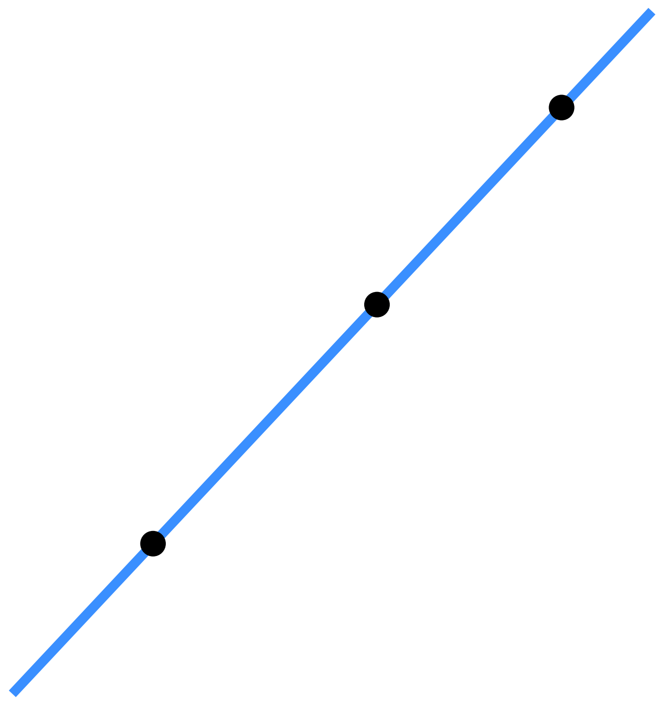

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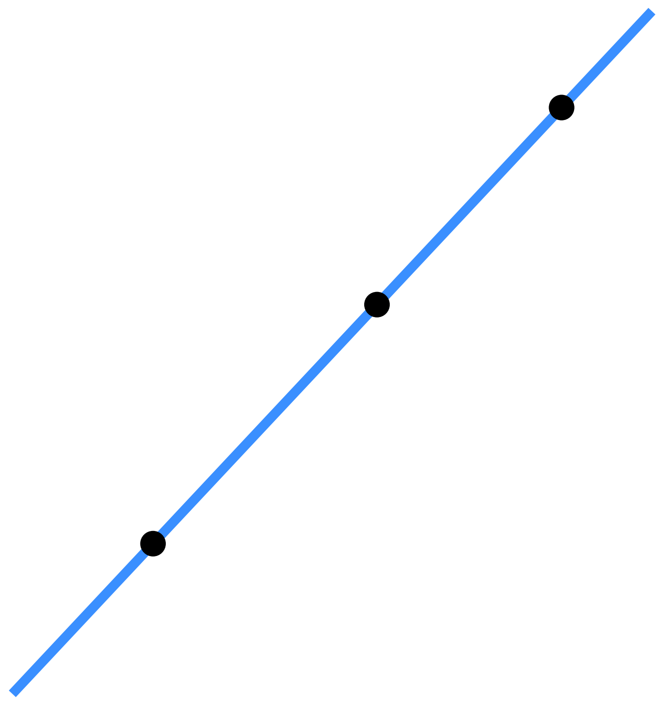

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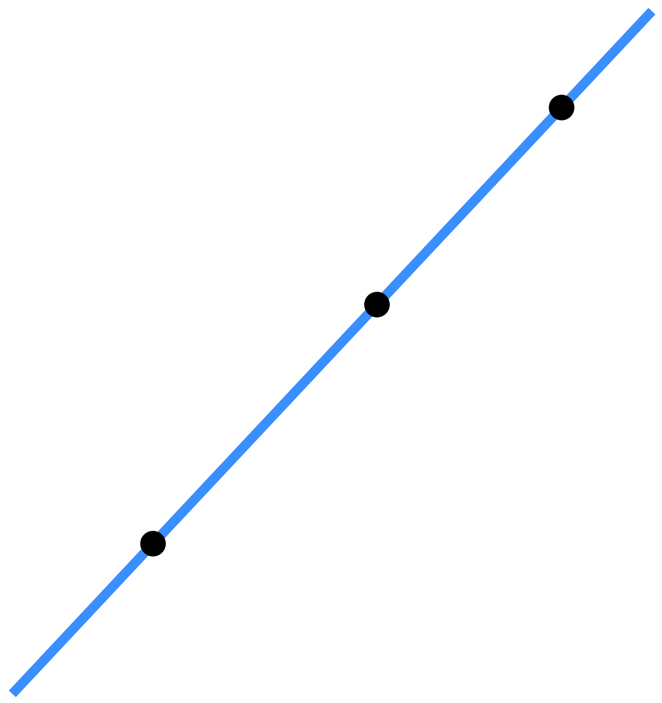
Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D}$ = set of disks

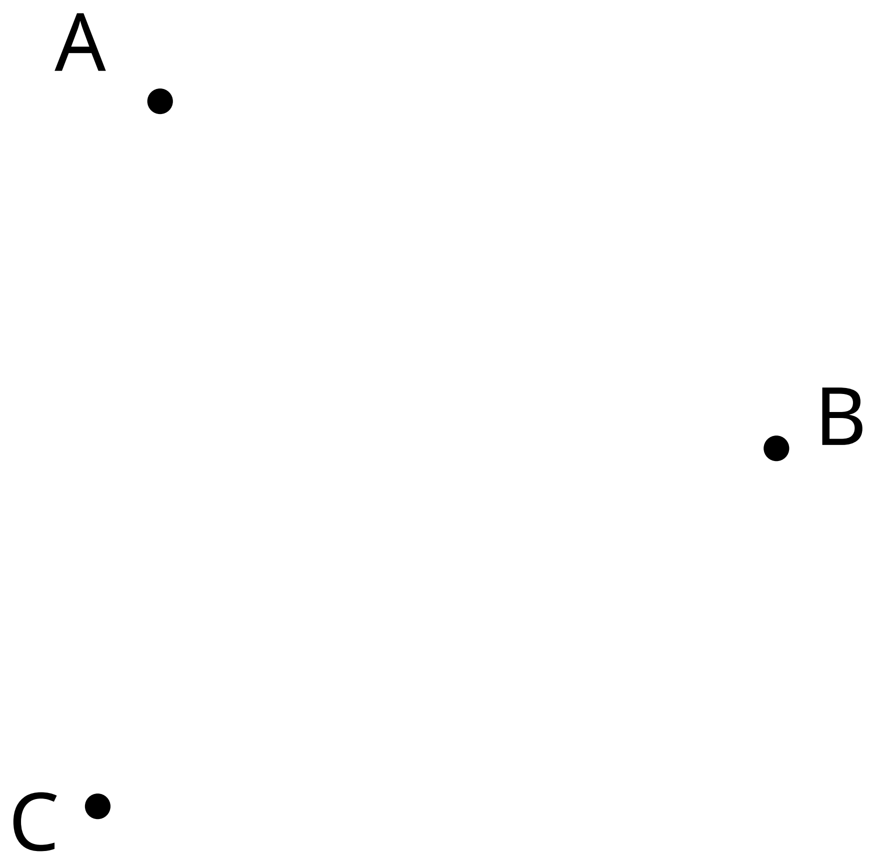


not shatter !

not relevant, since VC-dimension = maximum size of shattered subset

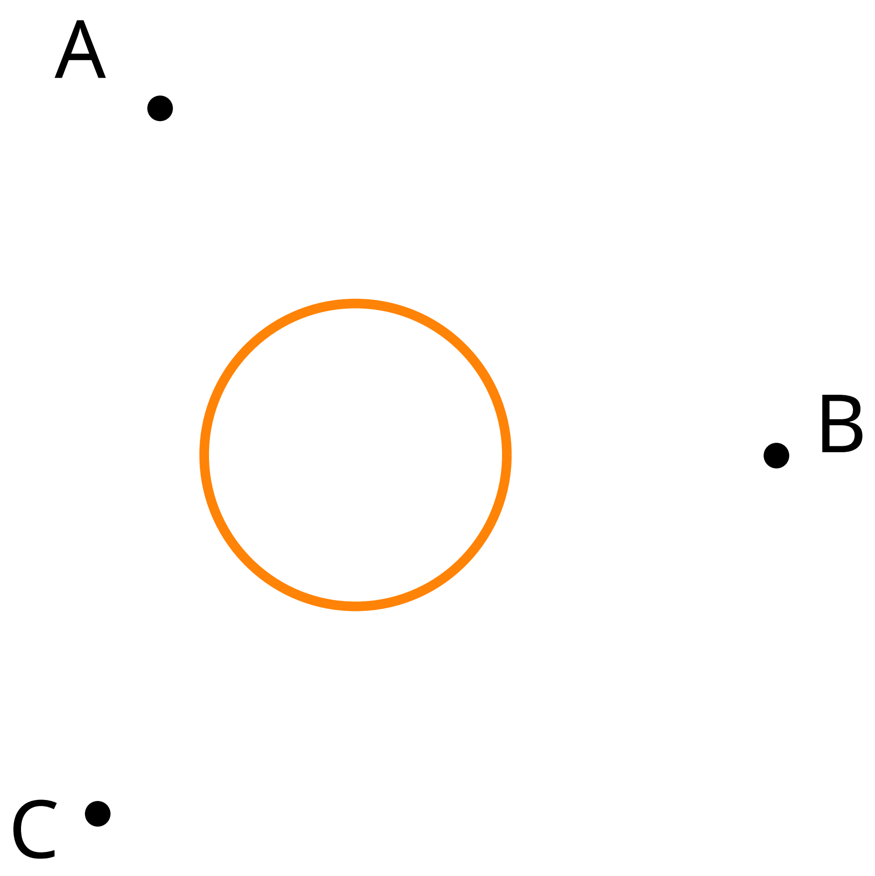
Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D} =$ set of disks



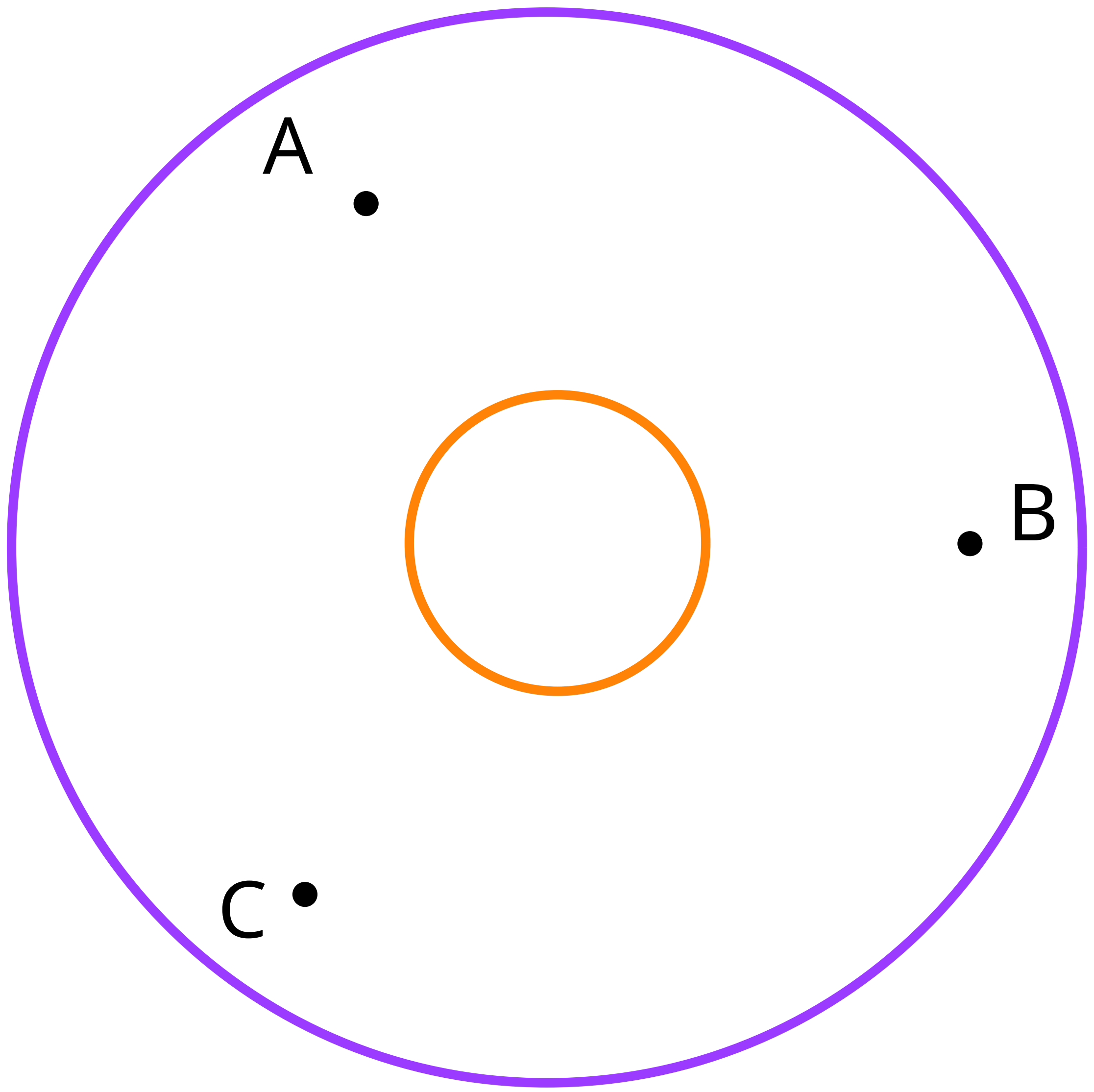
Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D}$ = set of disks



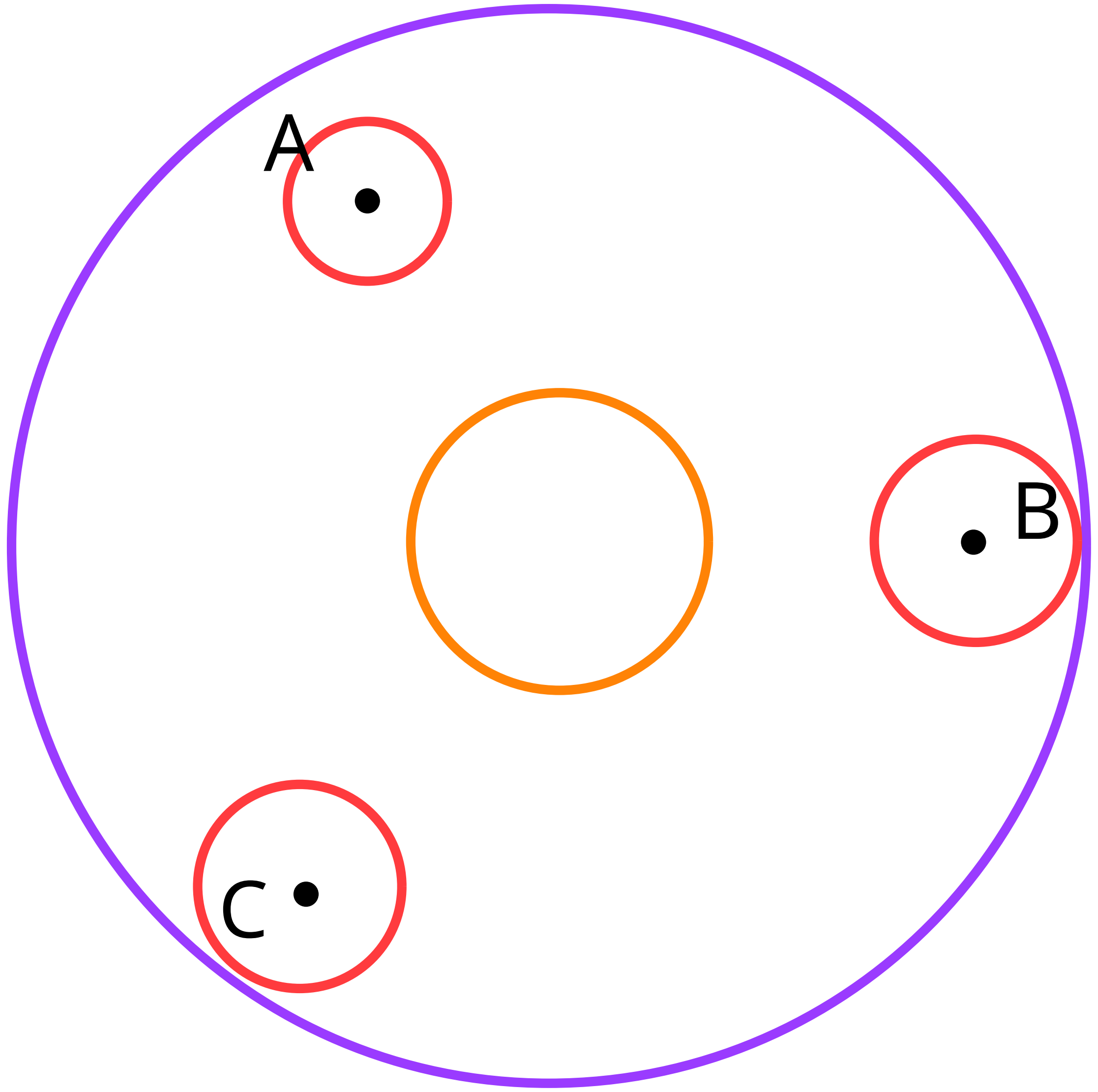
Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D}$ = set of disks



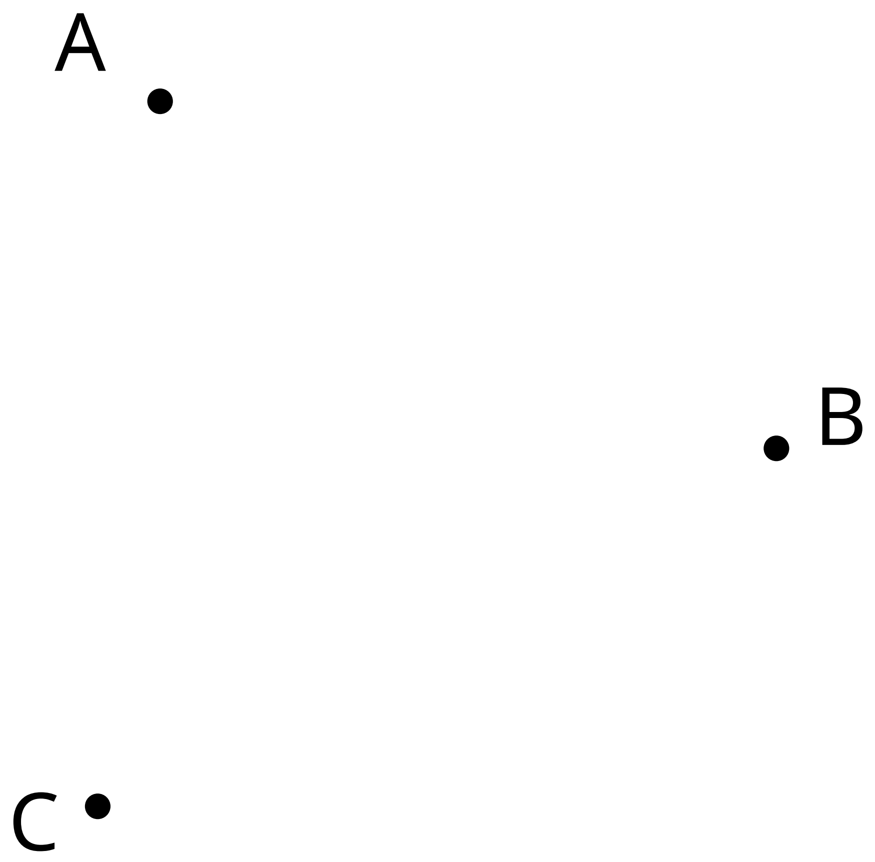
Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D} =$ set of disks



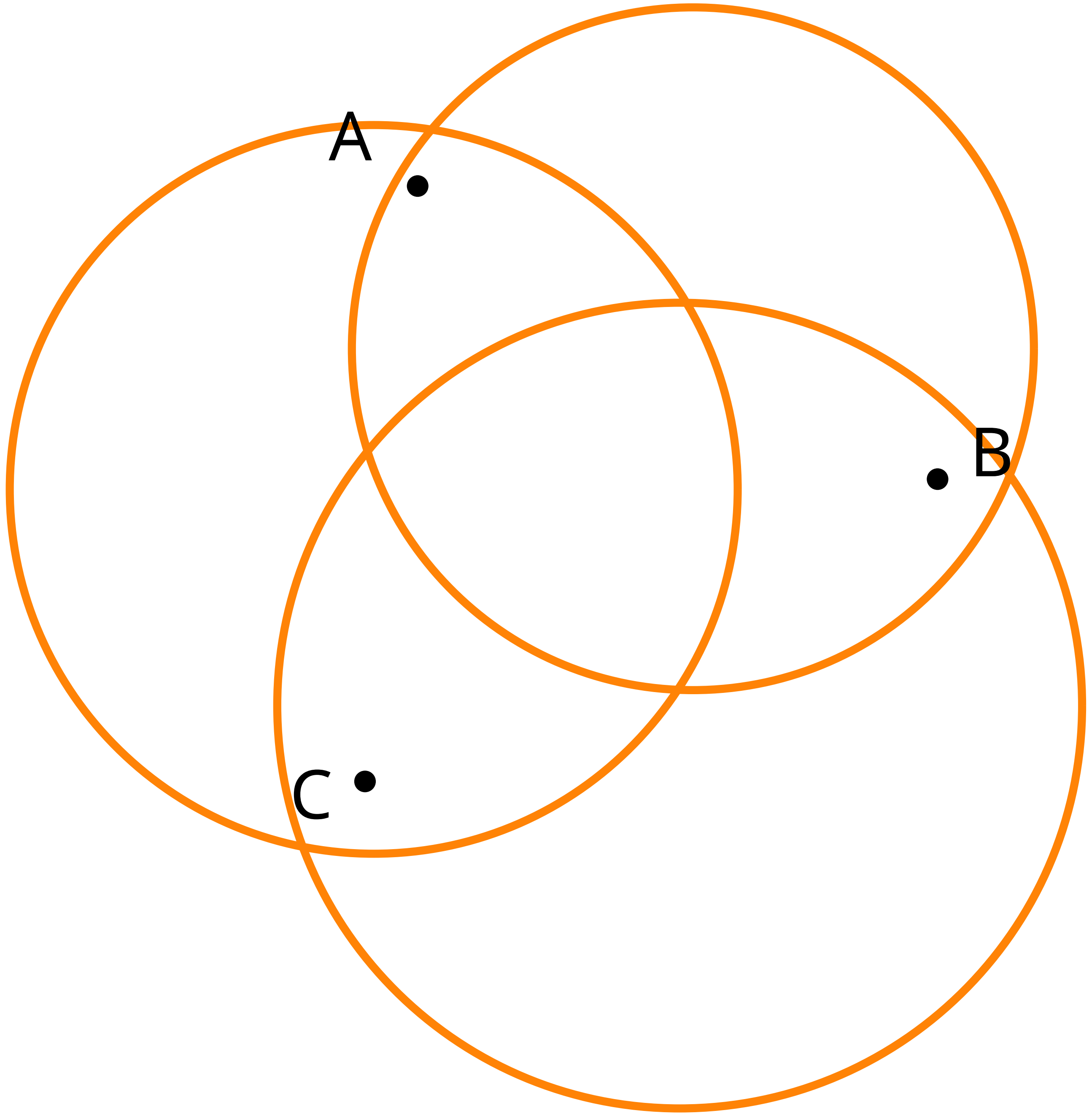
Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D} =$ set of disks



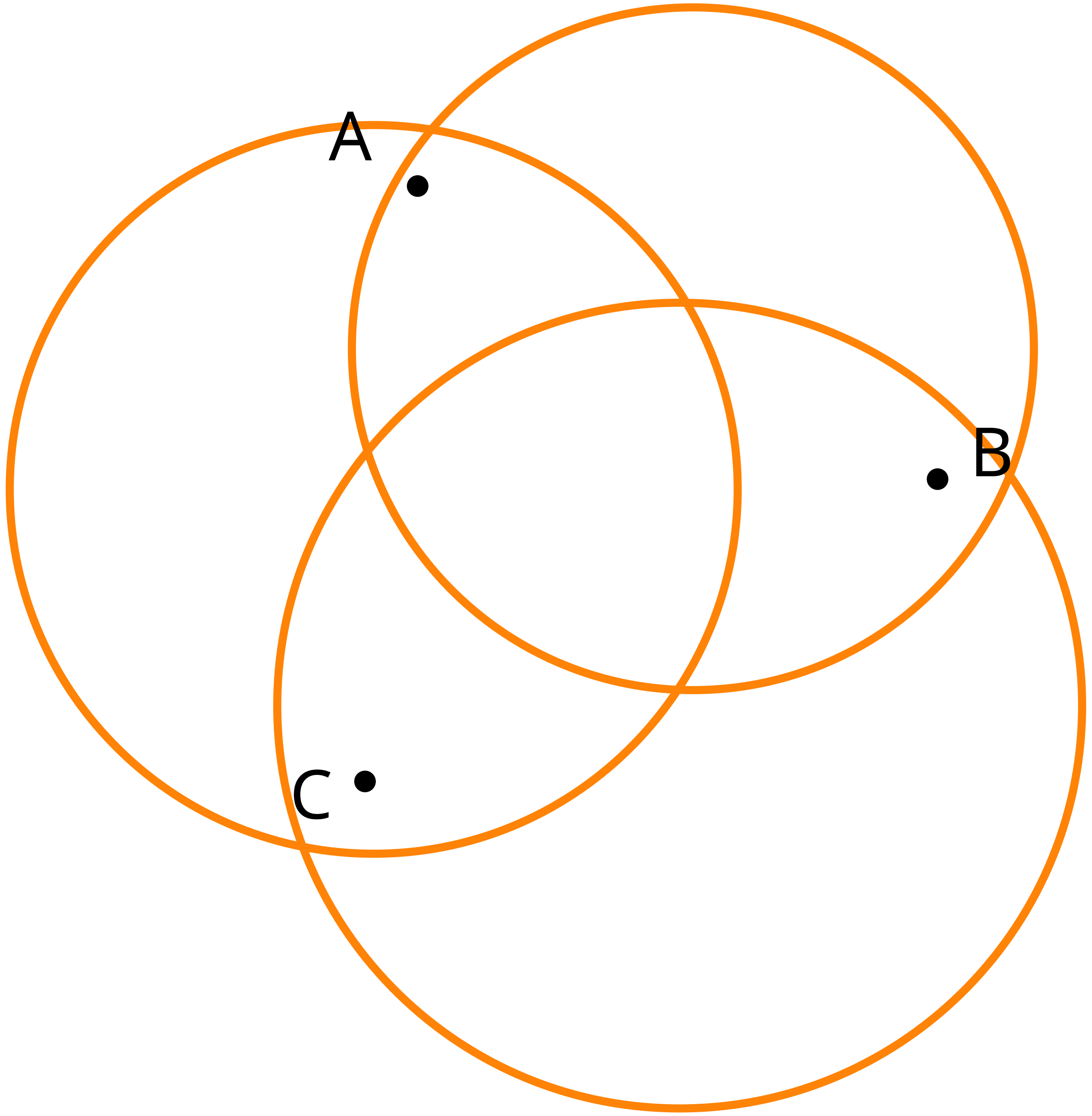
Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D} =$ set of disks



Example: disks as ranges

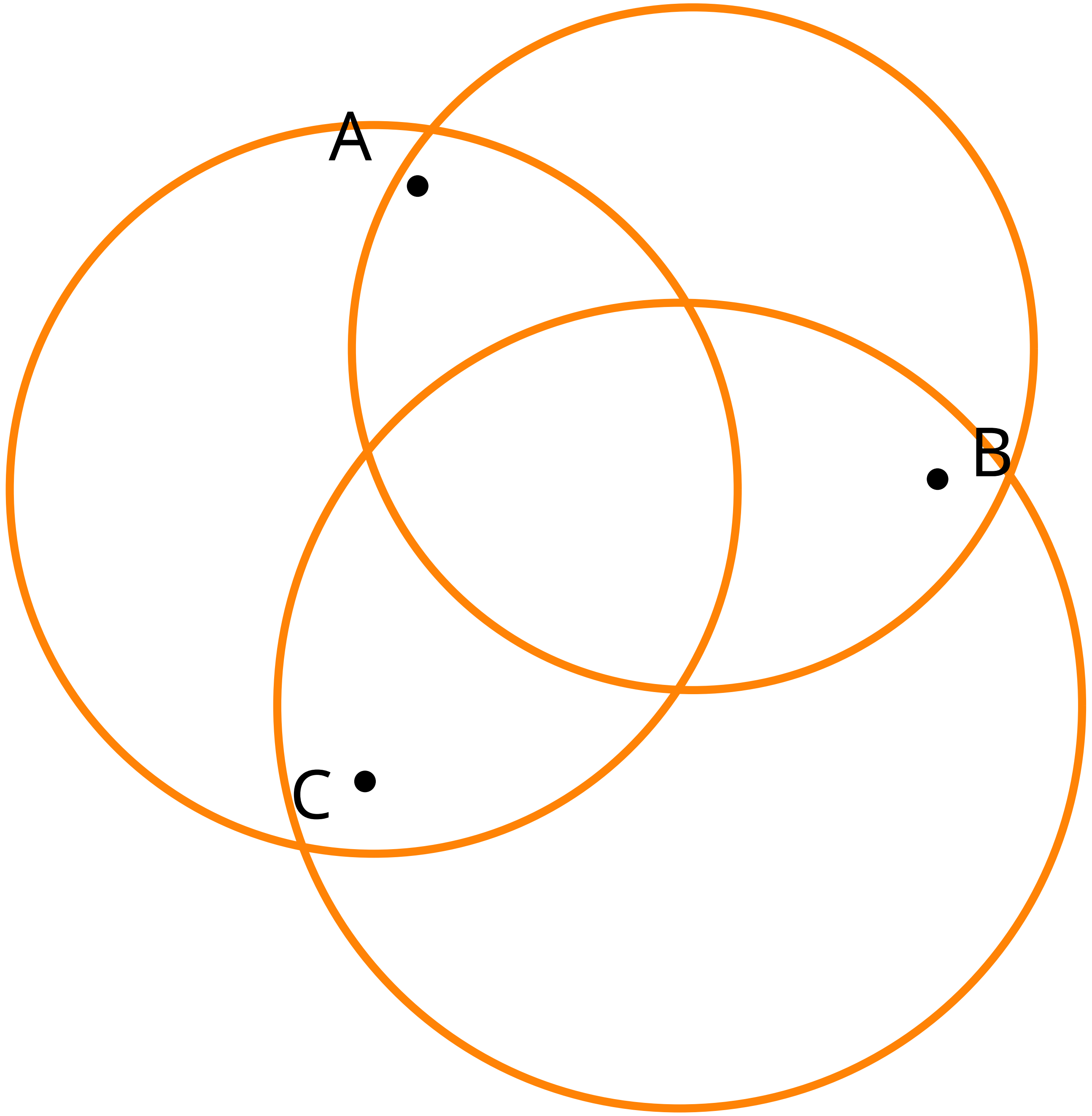
range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D}$ = set of disks



shatter !

Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D} =$ set of disks

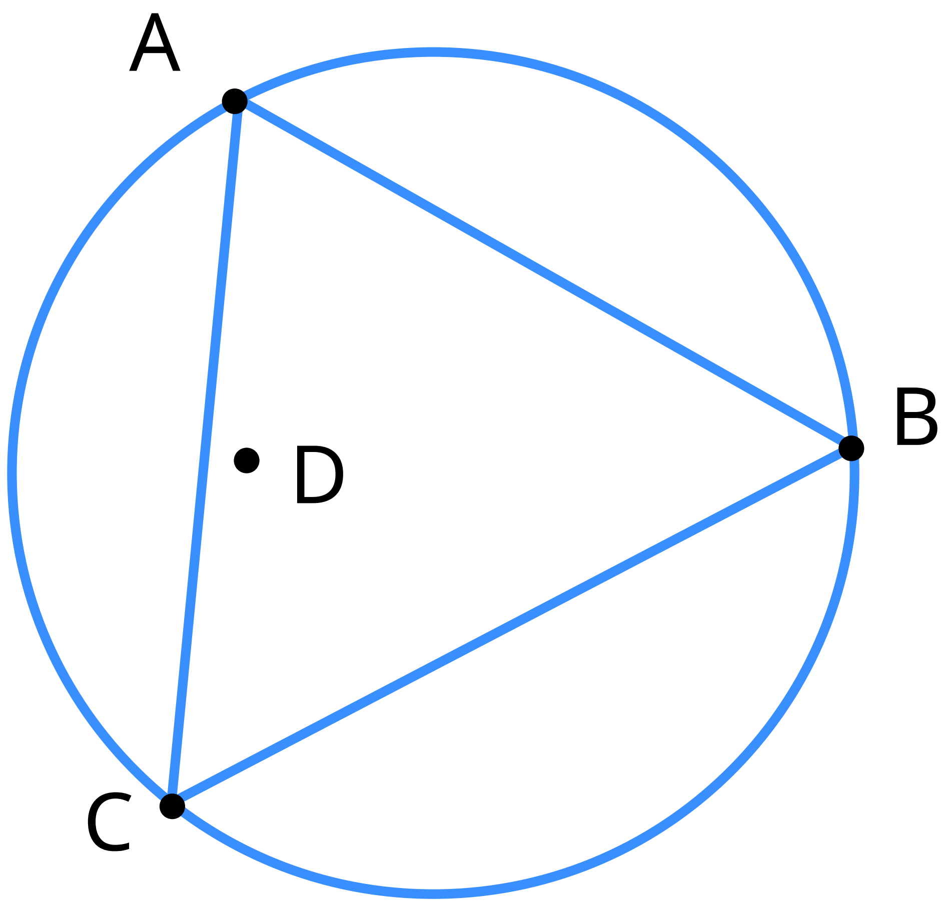


$\Rightarrow$ VC-dimension ≥ 3

shatter !

Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D}$ = set of disks

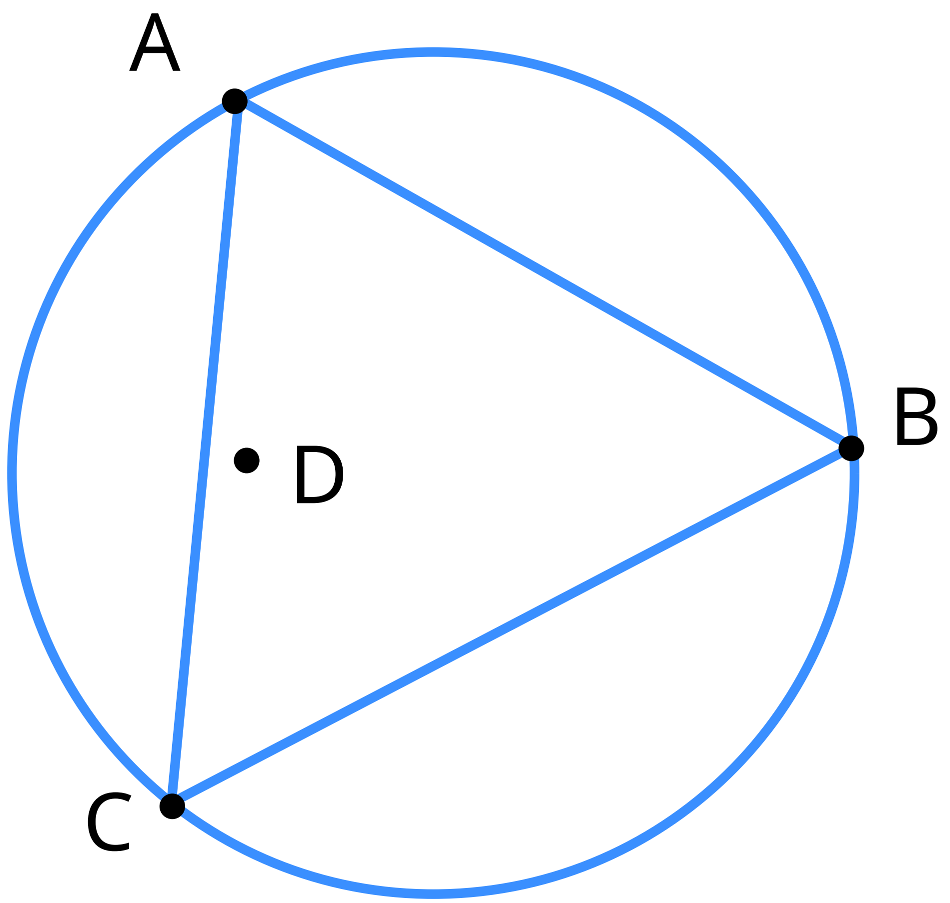


4 points

case 1: $D \in$ triangle(ABC)

Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D} =$ set of disks



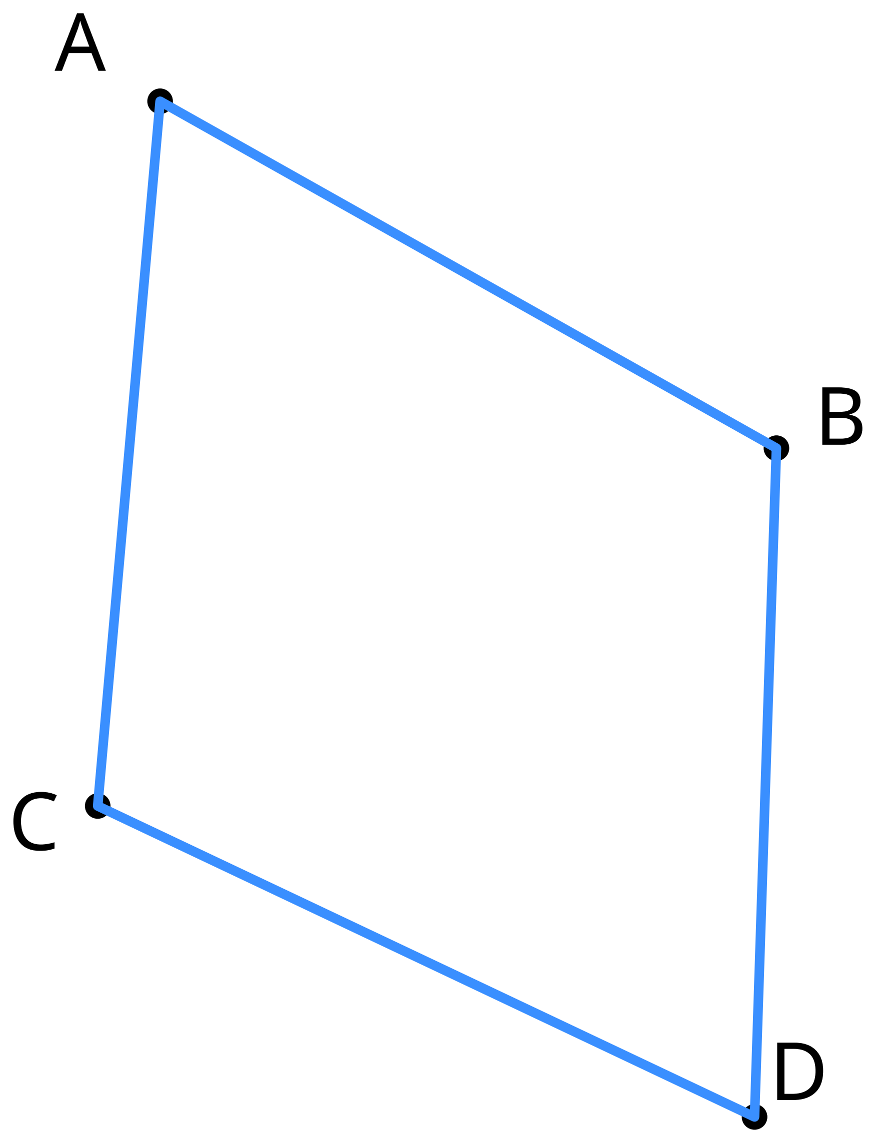
4 points

case 1: $D \in$ triangle(ABC)

not shatter !

Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D} =$ set of disks



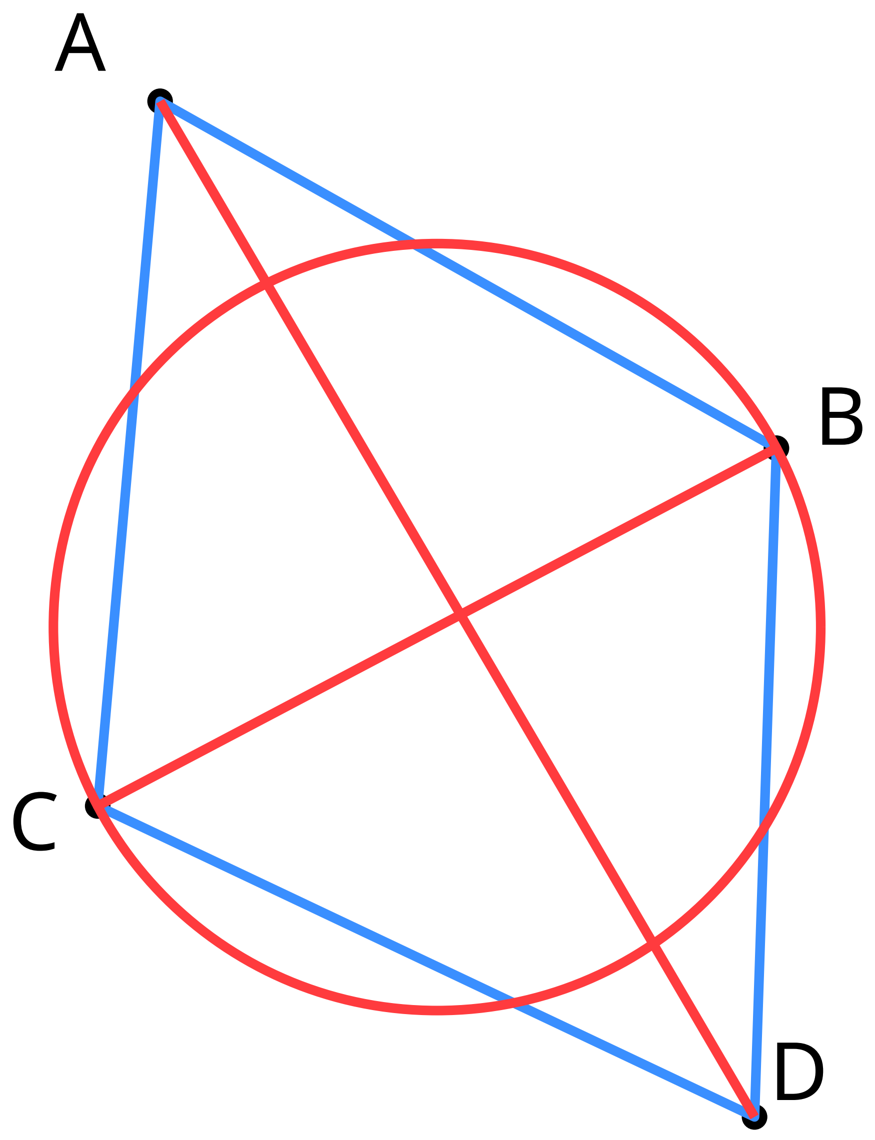
4 points

case 1: $D \in$ triangle(ABC)

case 2: $ABCD$ convex quadrilateral

Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D}$ = set of disks



4 points

case 1: $D \in$ triangle(ABC)

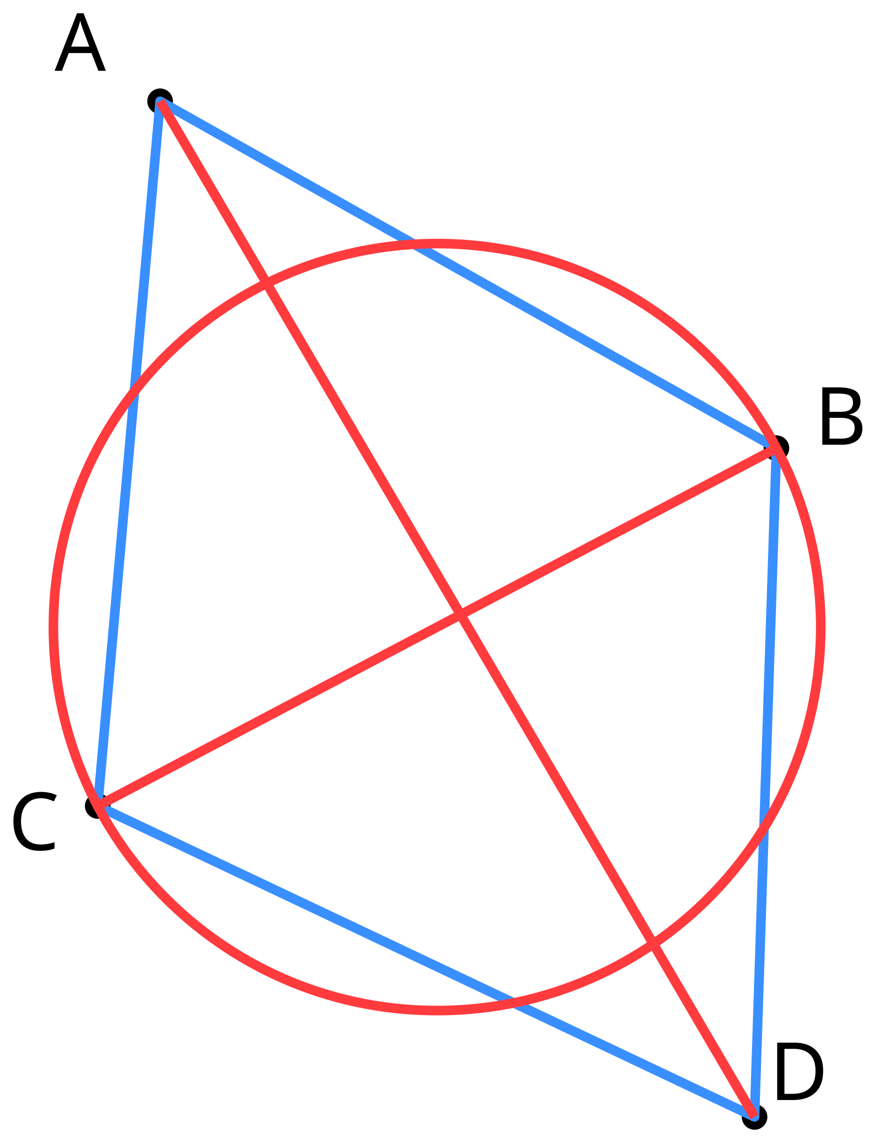
case 2: $ABCD$ convex quadrilateral

without proof:

can't get $\{A, D\}$ and $\{B, C\}$

Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D}$ = set of disks



4 points

case 1: $D \in$ triangle(ABC)

case 2: $ABCD$ convex quadrilateral

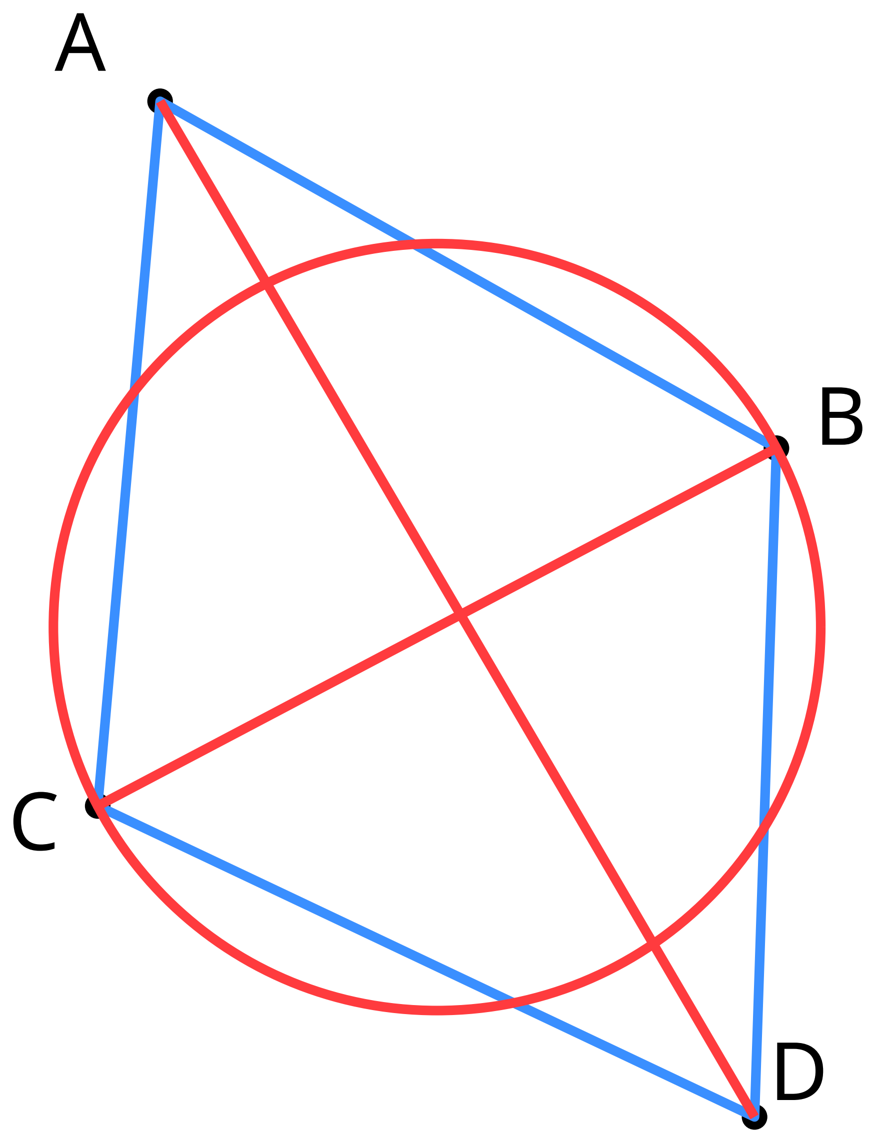
without proof:

can't get $\{A, D\}$ and $\{B, C\}$

not shatter !

Example: disks as ranges

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D}$ = set of disks



4 points

case 1: $D \in$ triangle(ABC)

case 2: $ABCD$ convex quadrilateral

without proof:

can't get $\{A, D\}$ and $\{B, C\}$

$\Rightarrow$ VC-dimension $= 3$

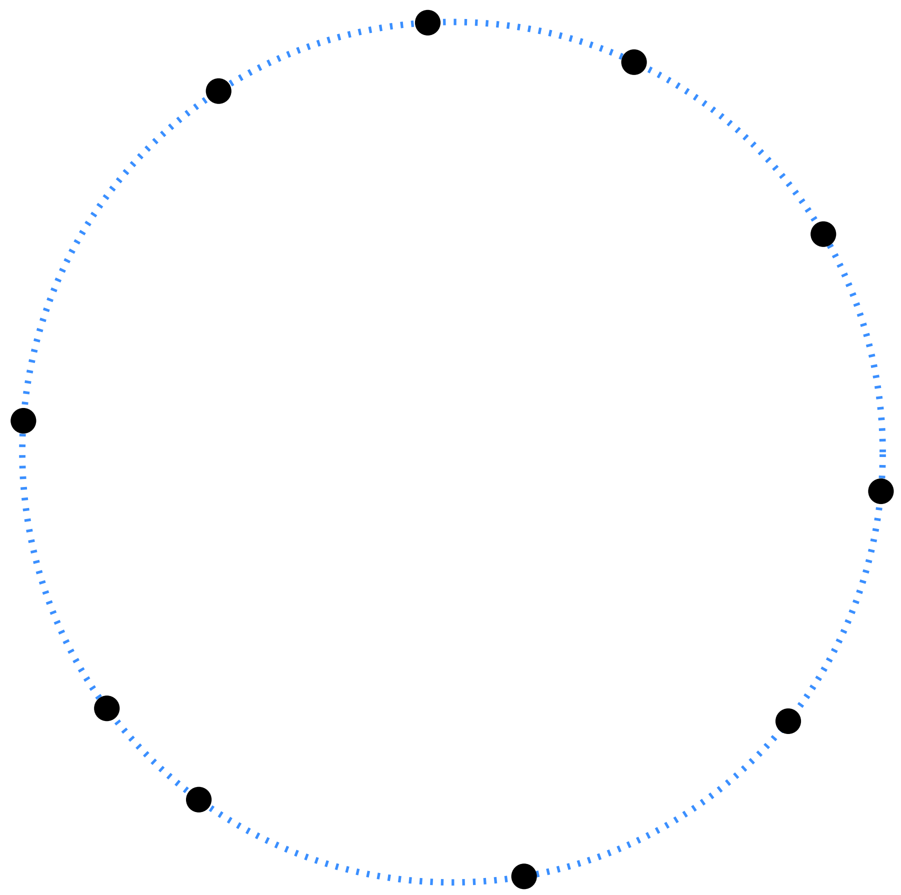
not shatter !

Example: convex sets as ranges

range space $(\mathbb{R}^2, \mathcal{C})$, with $\mathcal{C} =$ set of closed convex sets

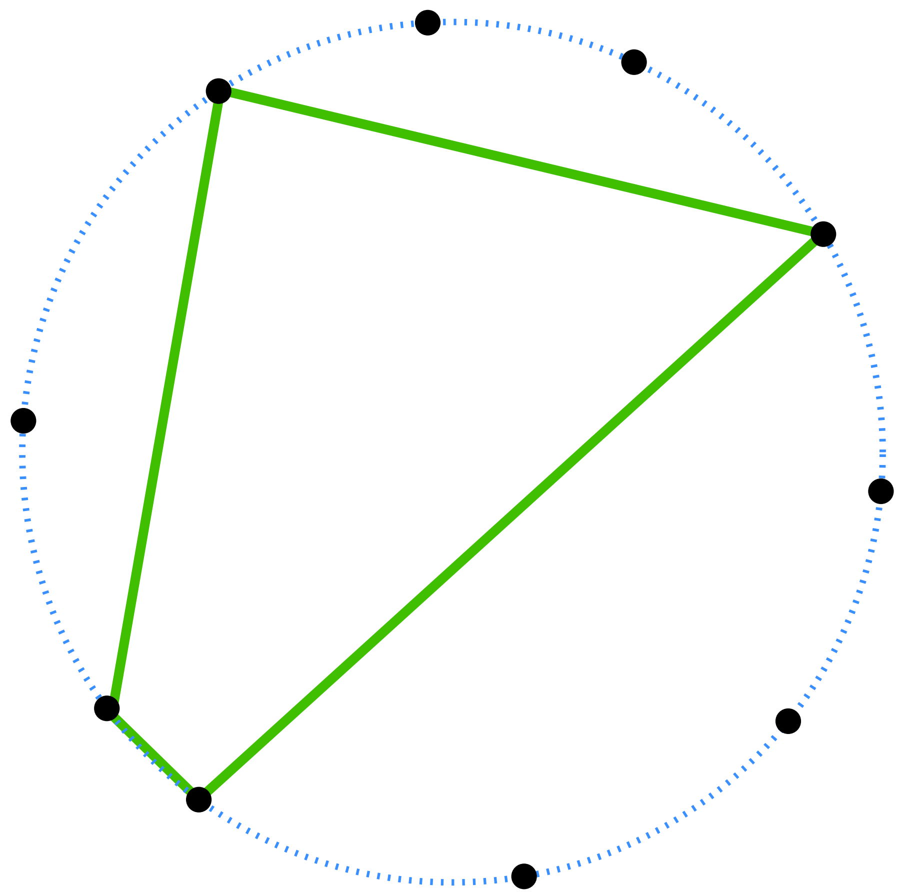
Example: convex sets as ranges

range space $(\mathbb{R}^2, \mathcal{C})$, with $\mathcal{C}$ = set of closed convex sets



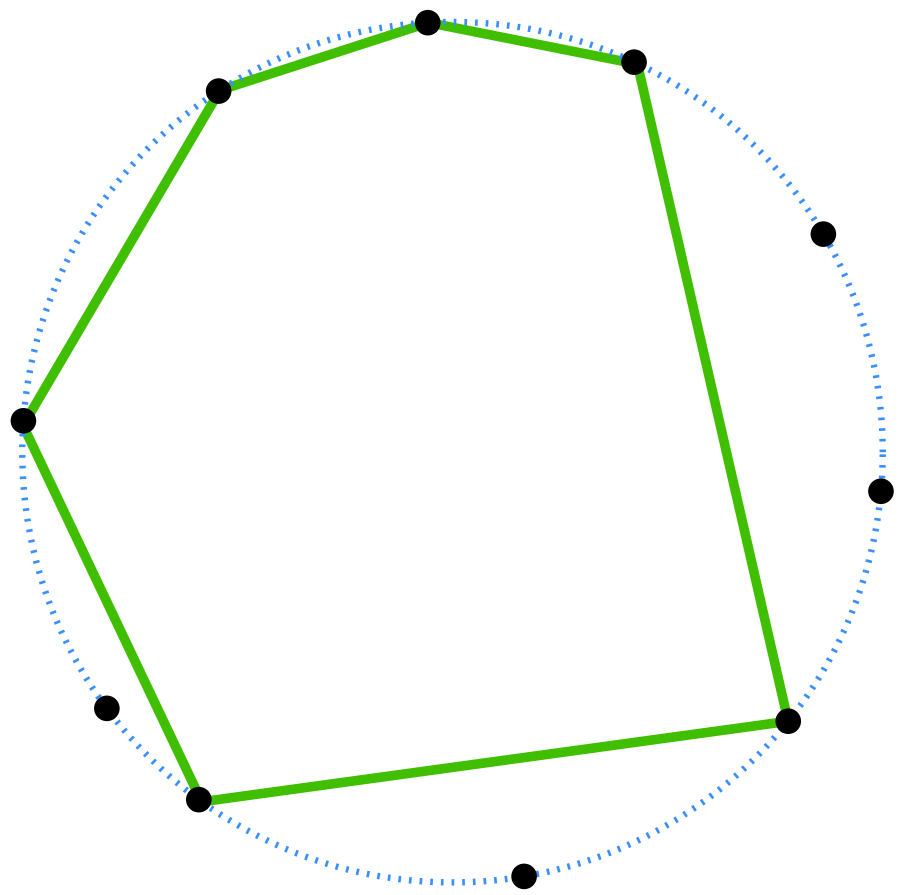
Example: convex sets as ranges

range space $(\mathbb{R}^2, \mathcal{C})$, with $\mathcal{C} =$ set of closed convex sets



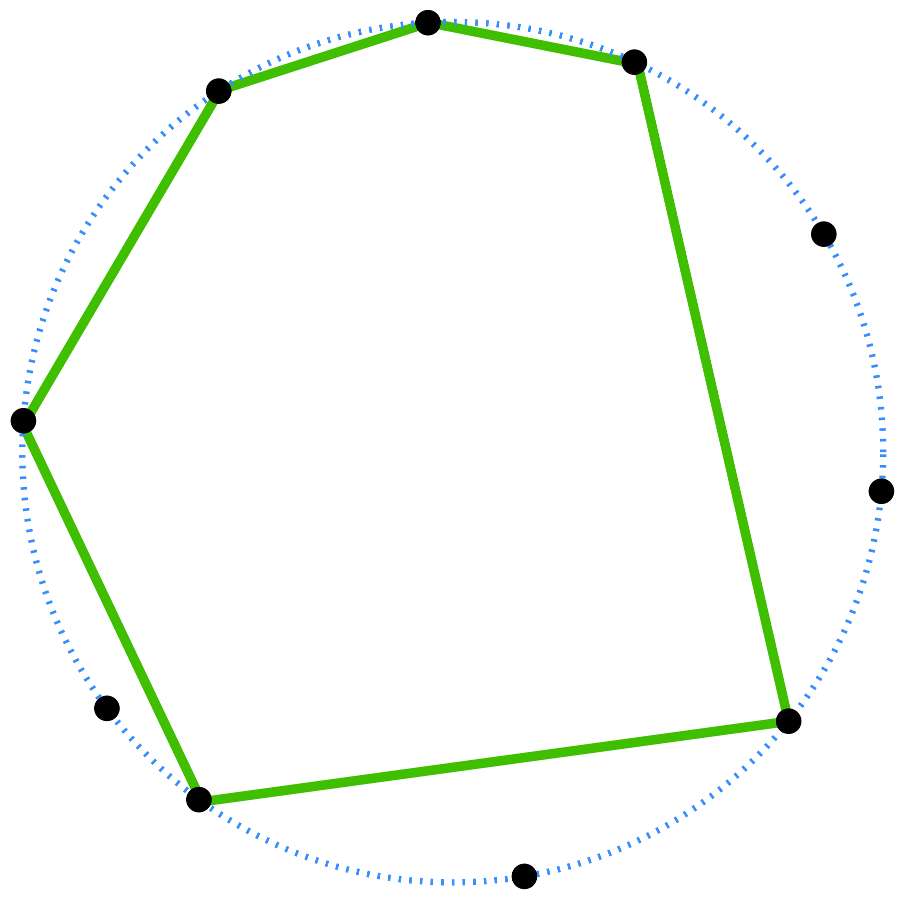
Example: convex sets as ranges

range space $(\mathbb{R}^2, \mathcal{C})$, with $\mathcal{C}$ = set of closed convex sets



Example: convex sets as ranges

range space $(\mathbb{R}^2, \mathcal{C})$, with $\mathcal{C} =$ set of closed convex sets



$\Rightarrow$ VC-dimension $= \infty$

Quiz

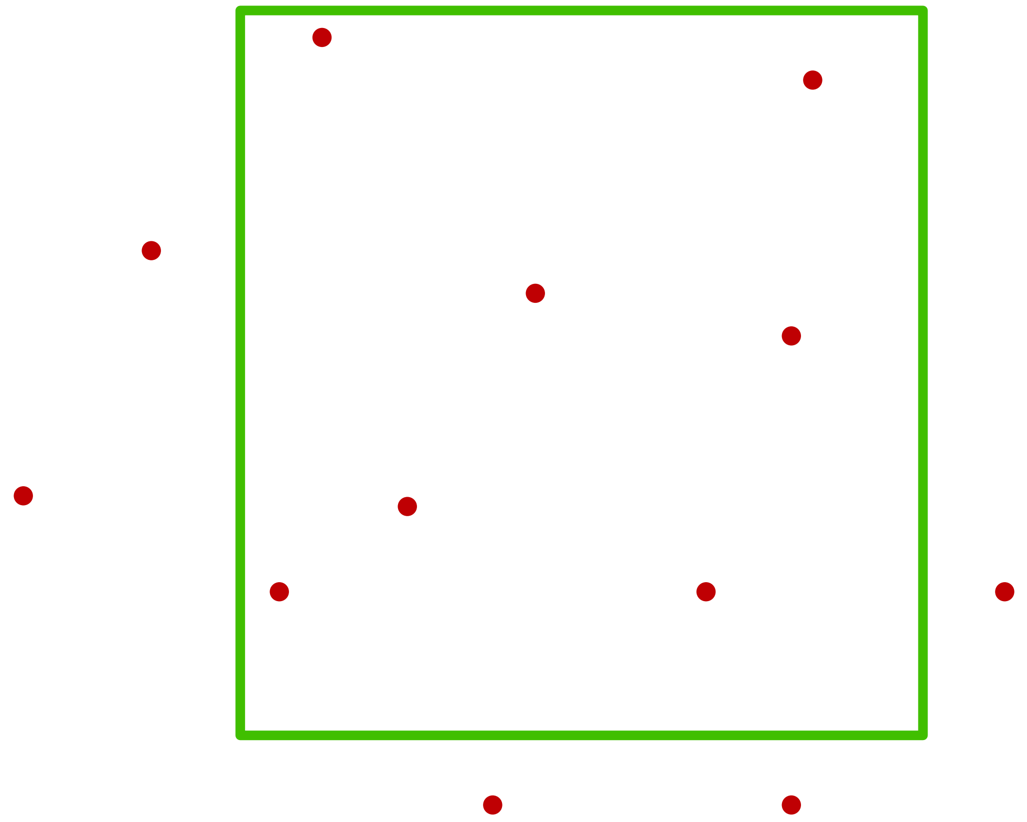
range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR} =$ set of axis-aligned rectangles

What is its VC-dimension?

A 4

B 5

C ∞



Quiz

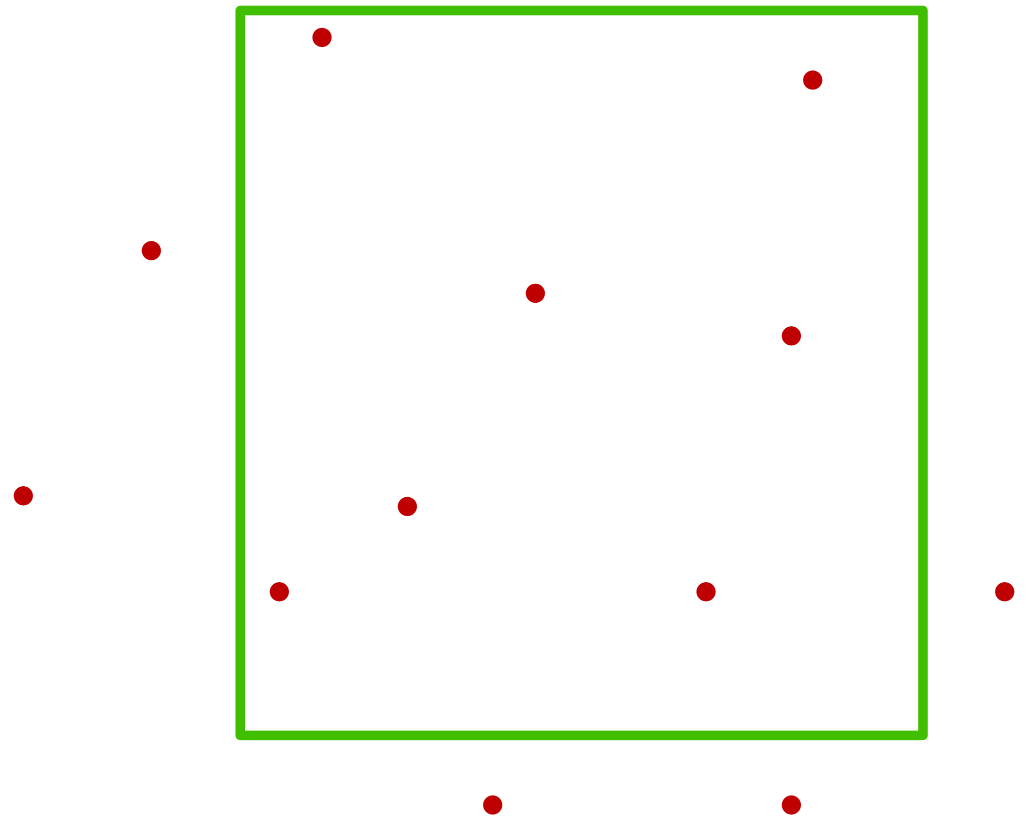
range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR} =$ set of axis-aligned rectangles

What is its VC-dimension?

A 4

B 5

C ∞



Example: rectangles as ranges

range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR} =$ set of axis-aligned rectangles

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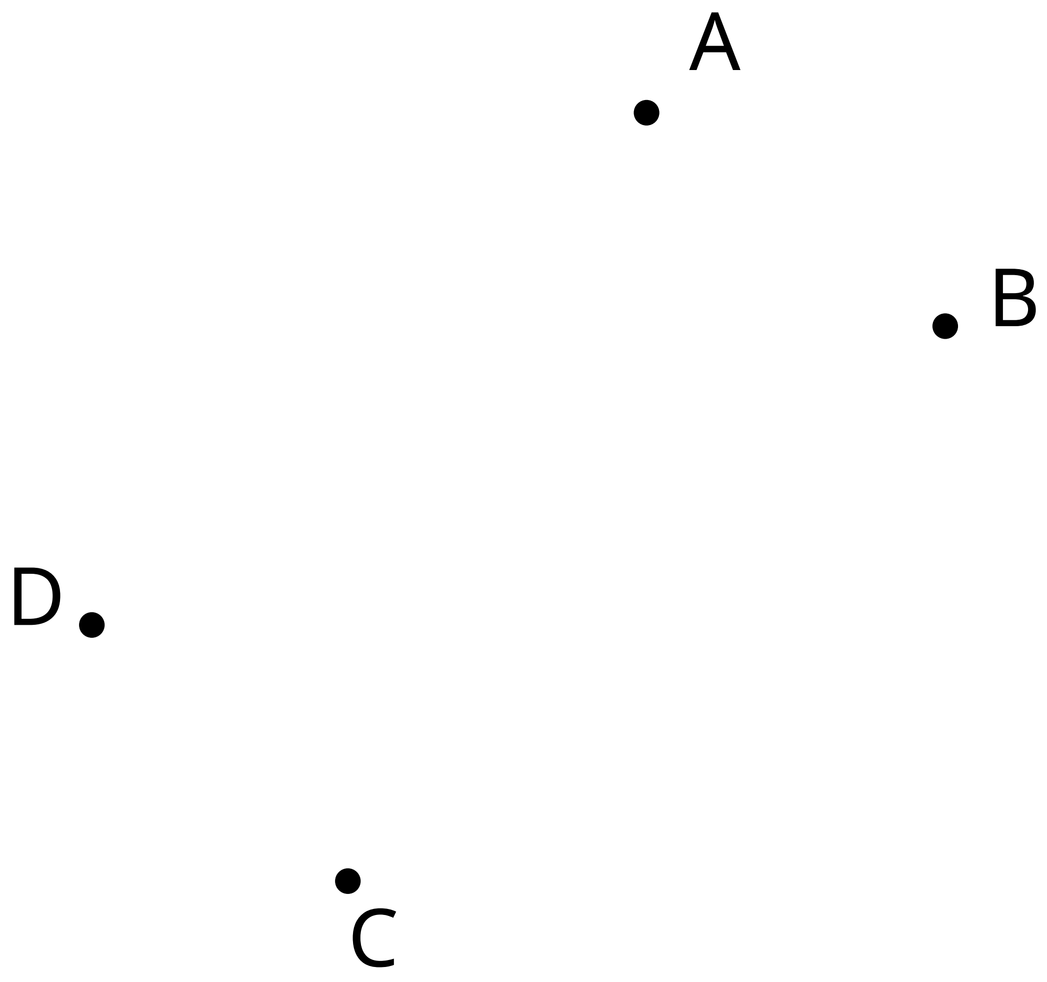
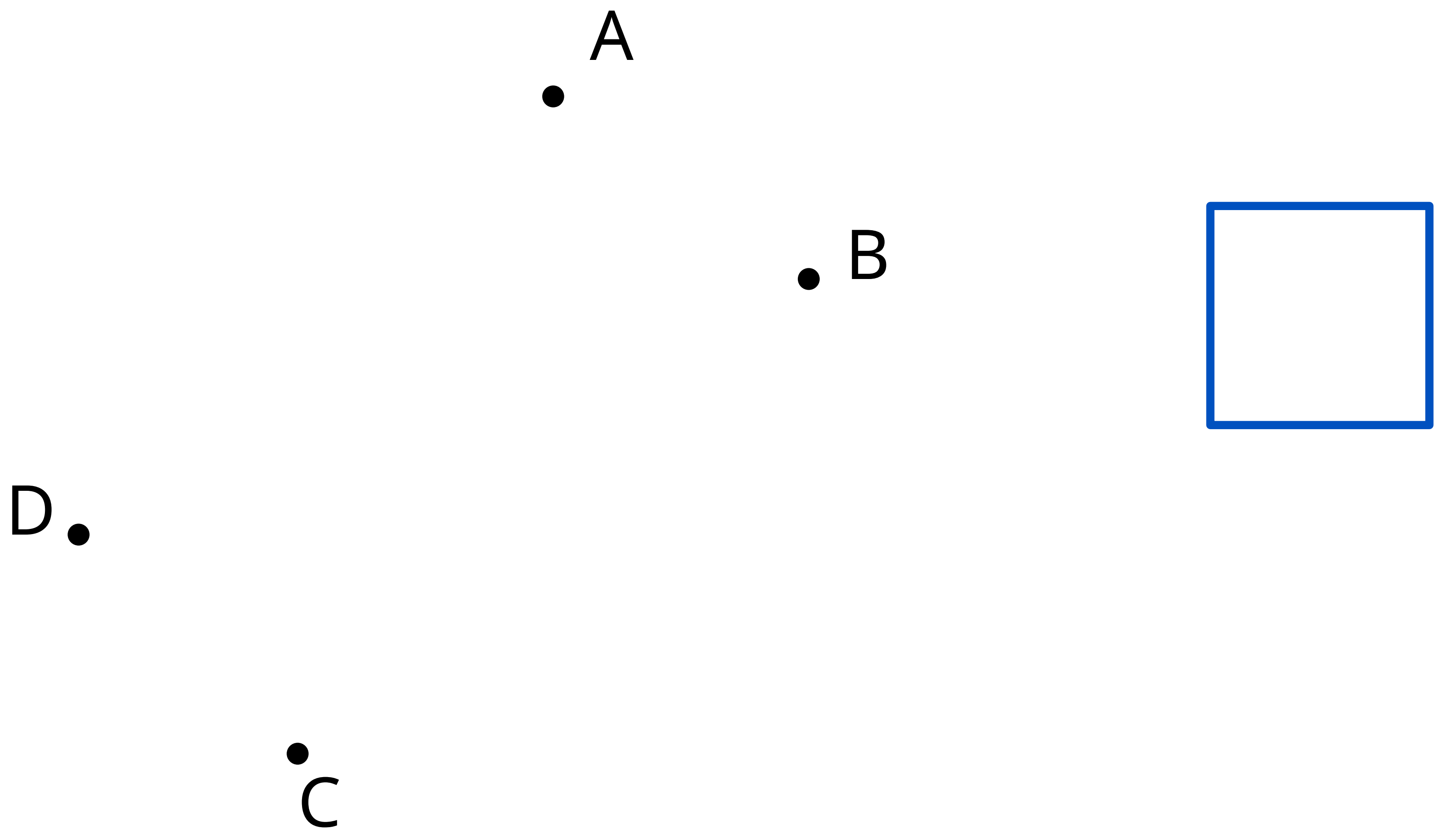
range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles



Example: rectangles as ranges

range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles



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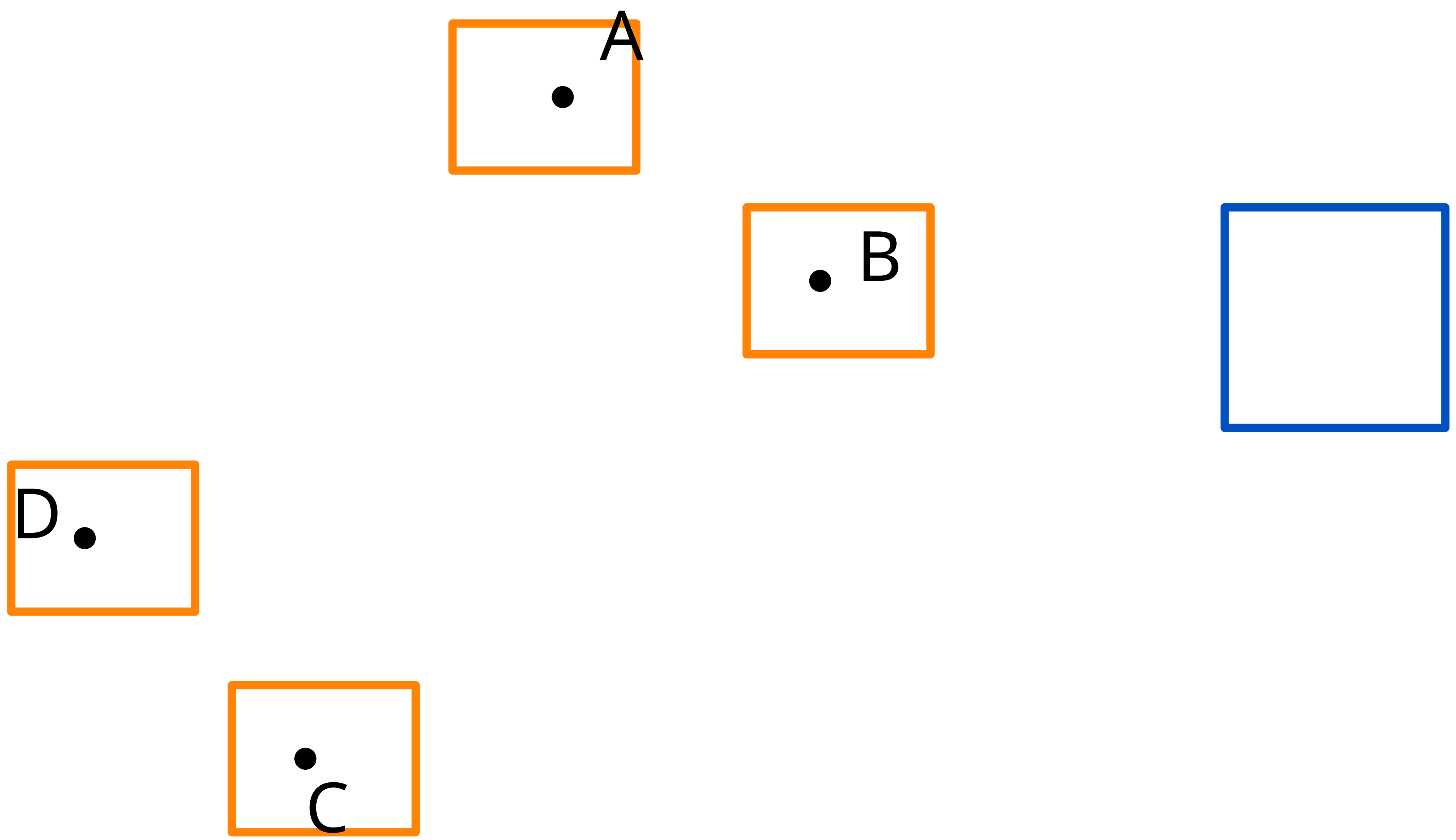
range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles



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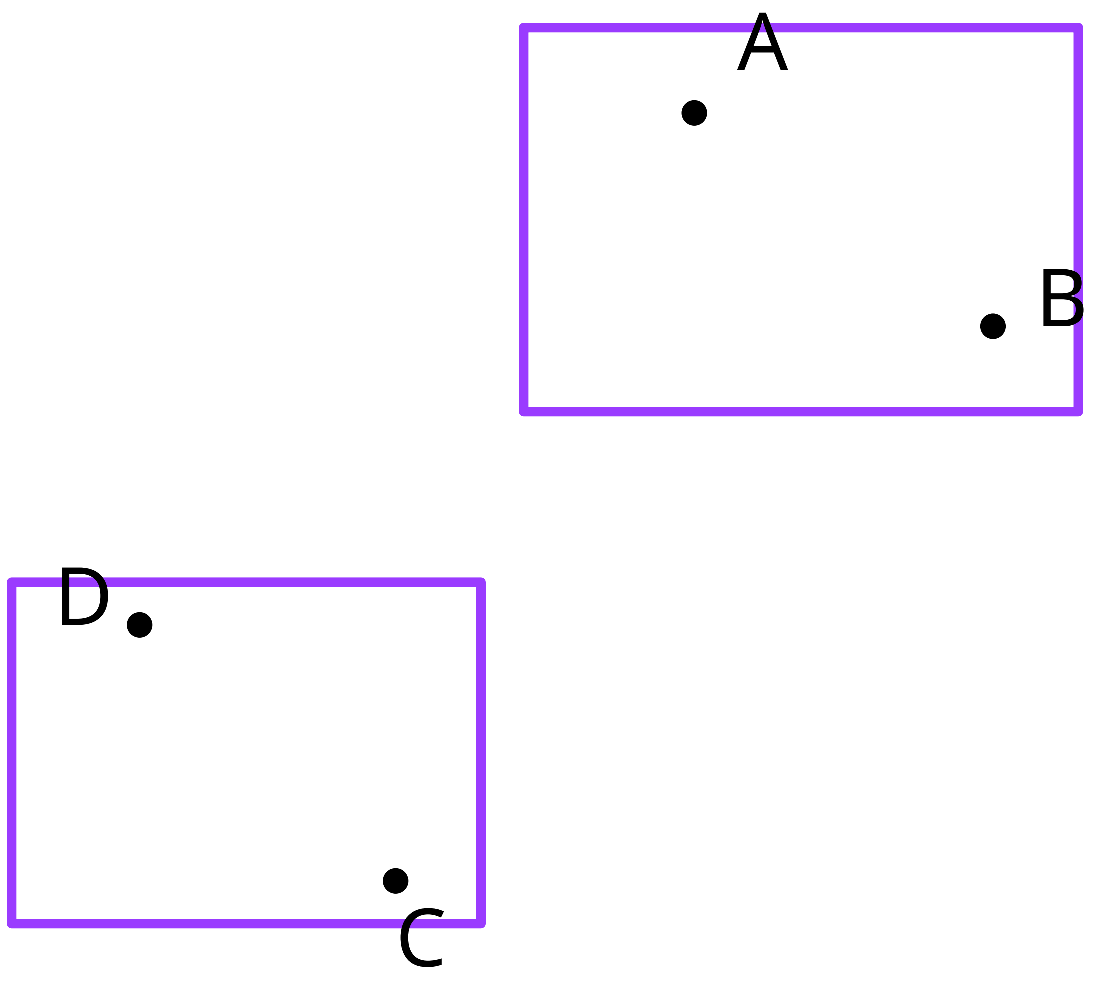

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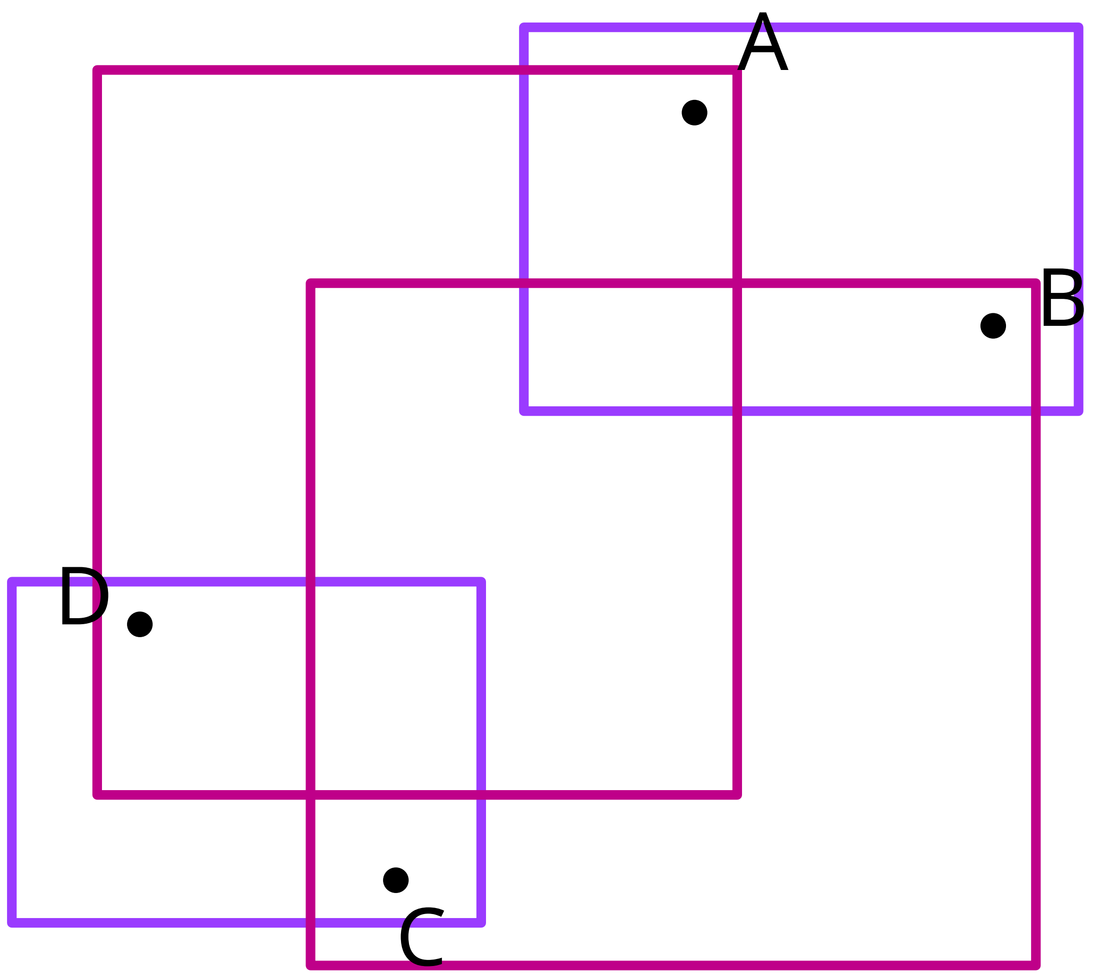

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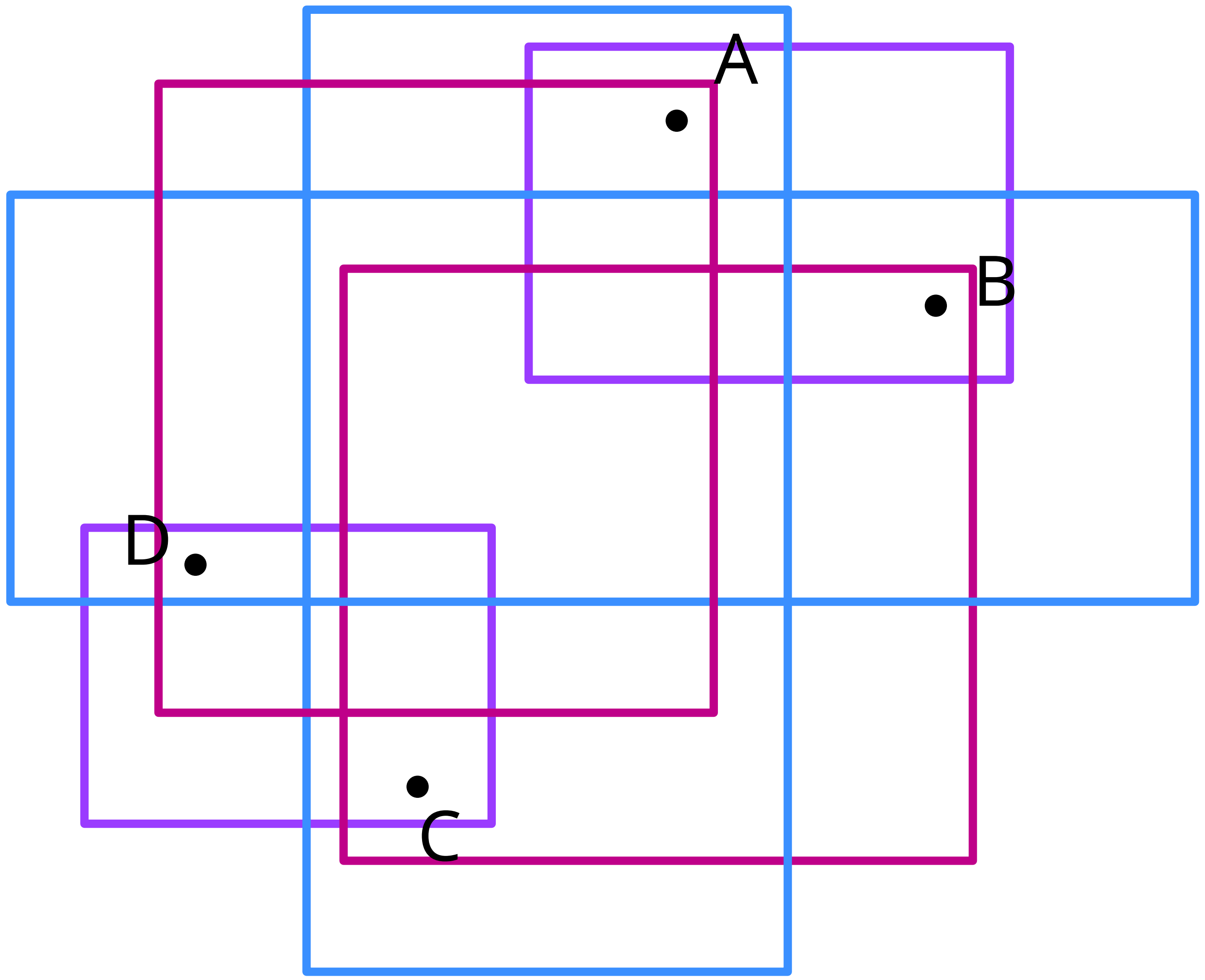

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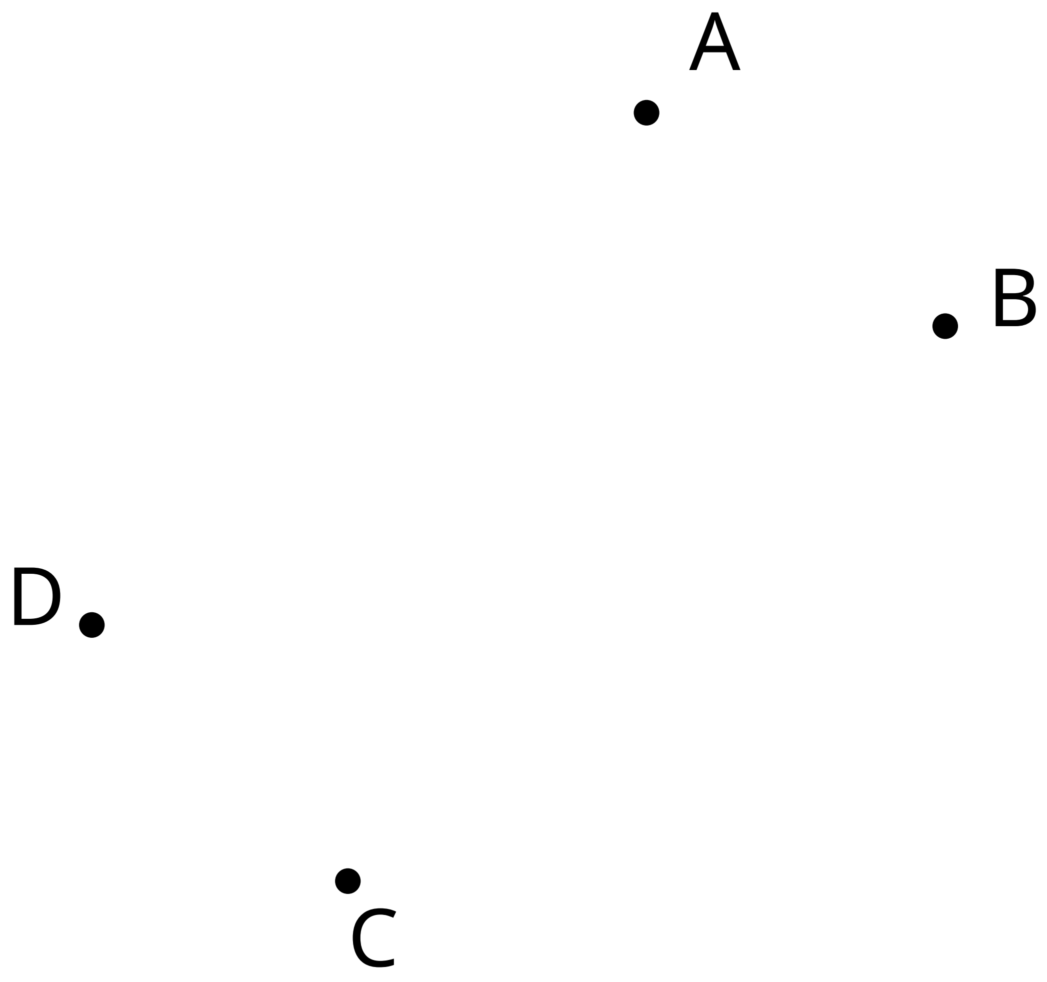
range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles



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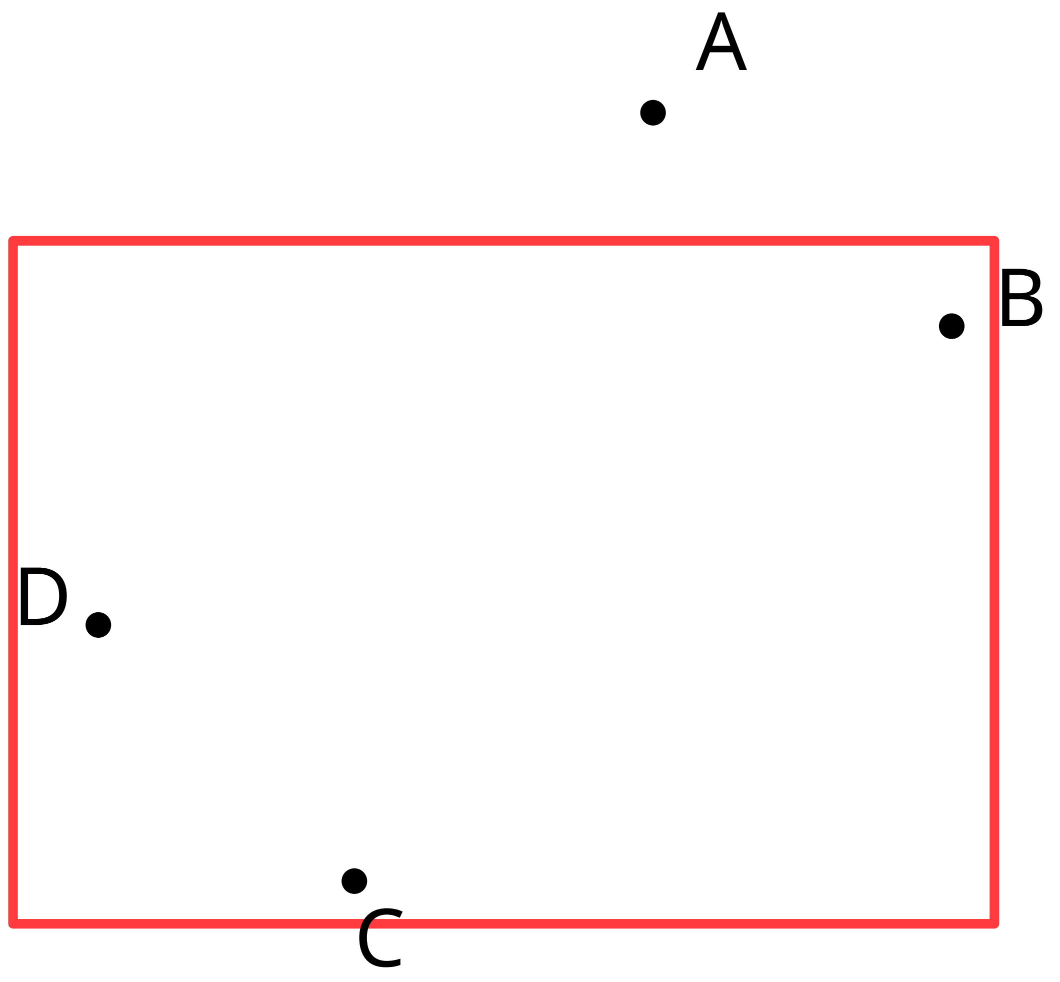

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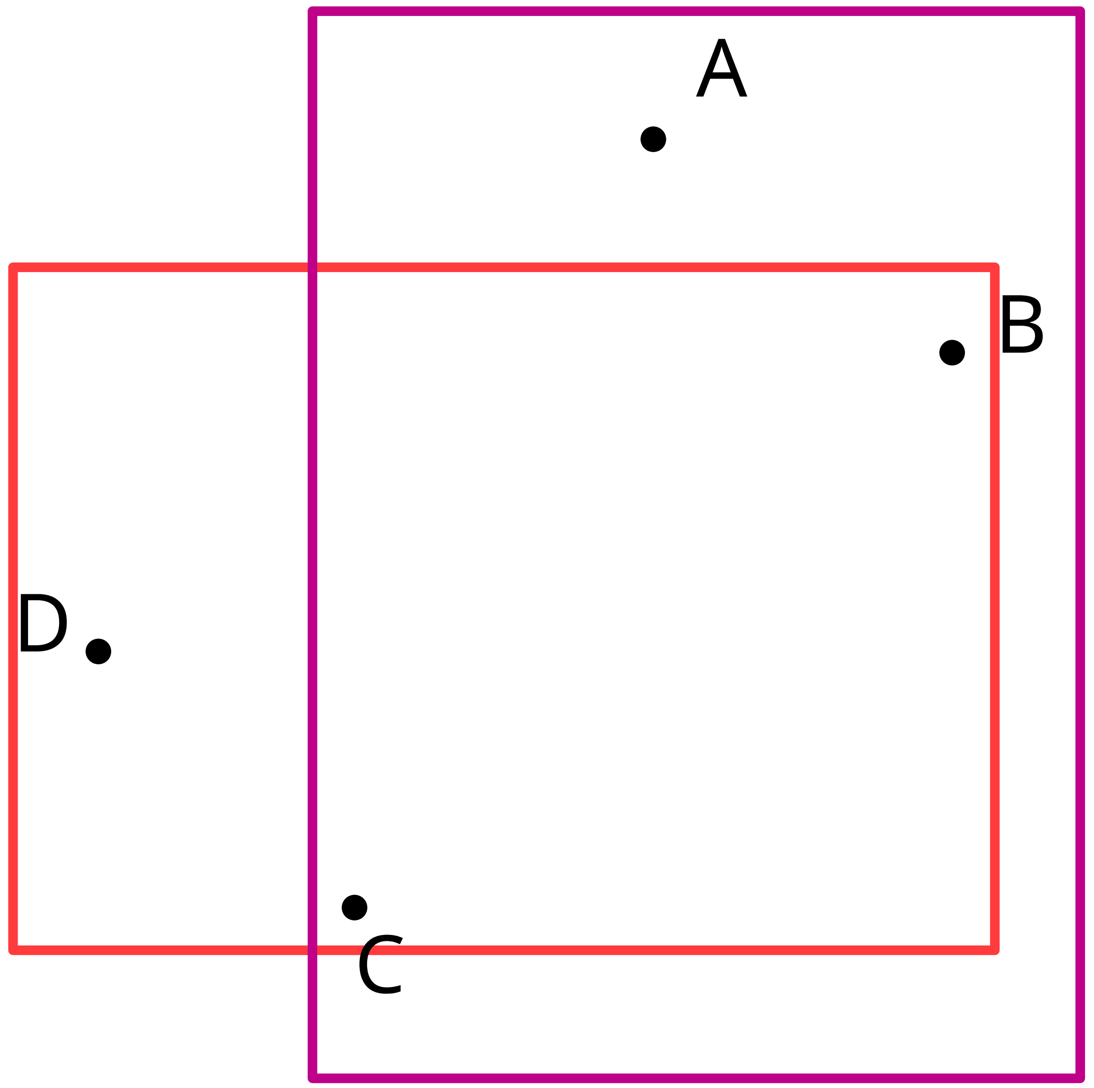

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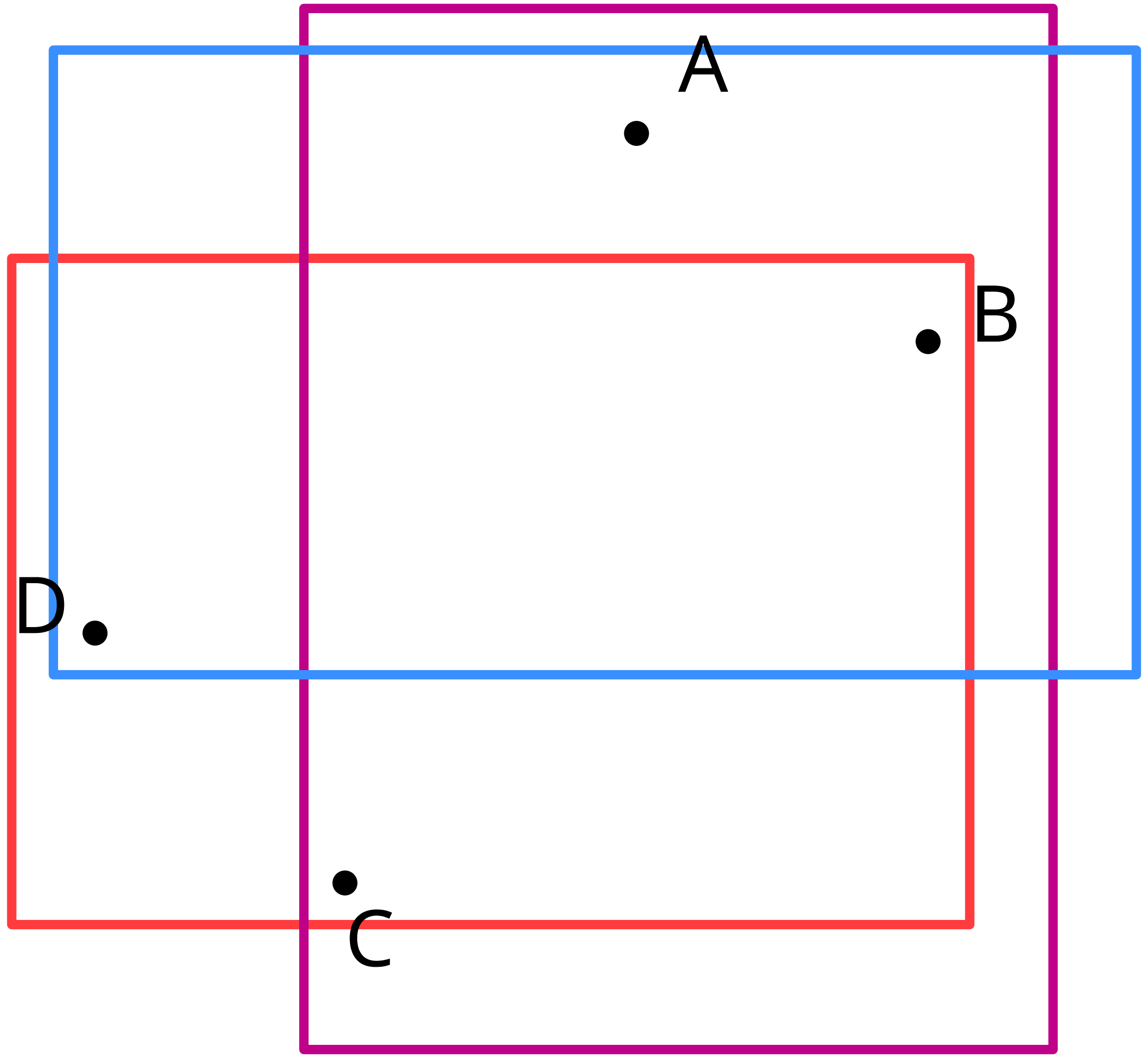

Example: rectangles as ranges

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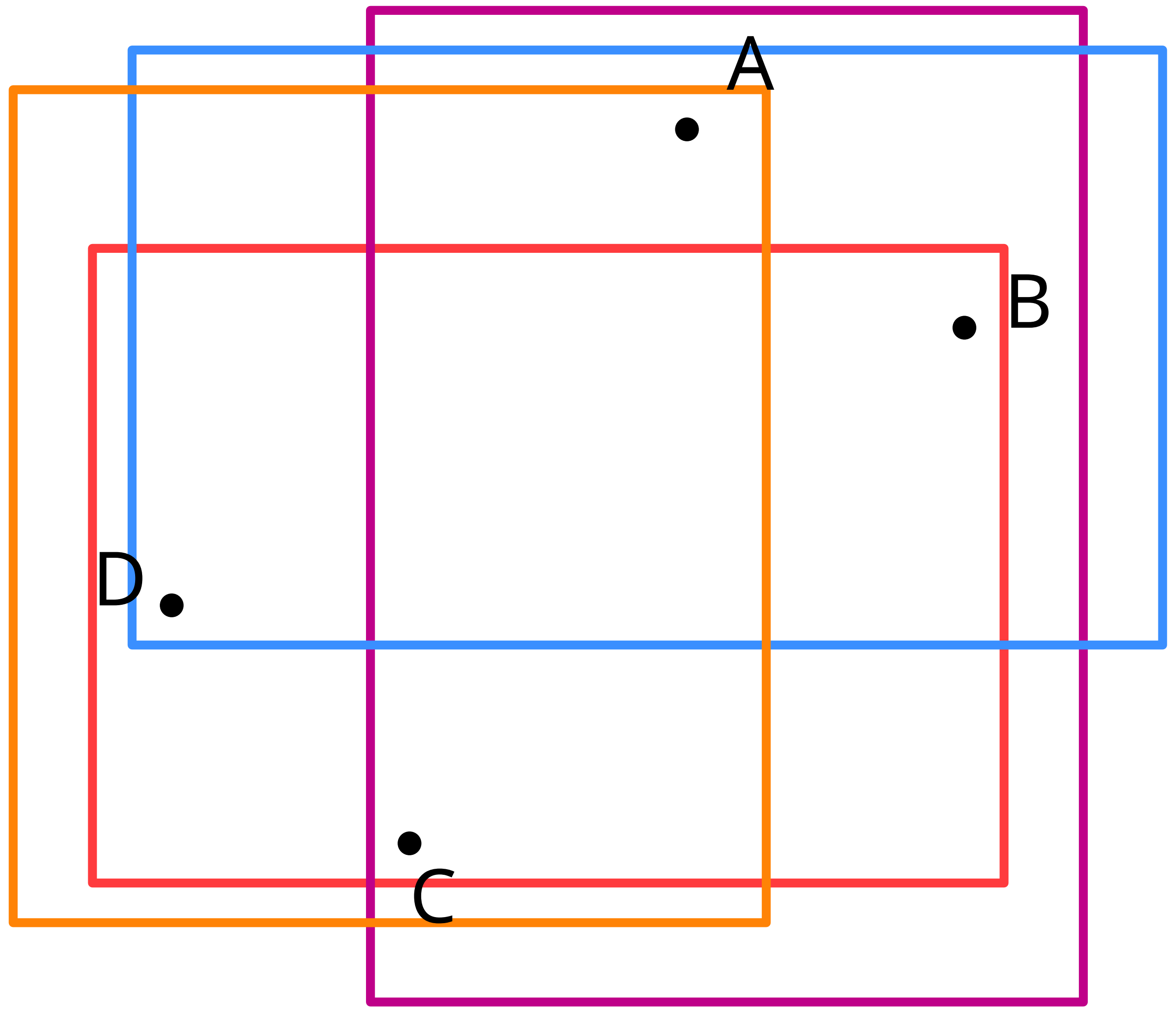
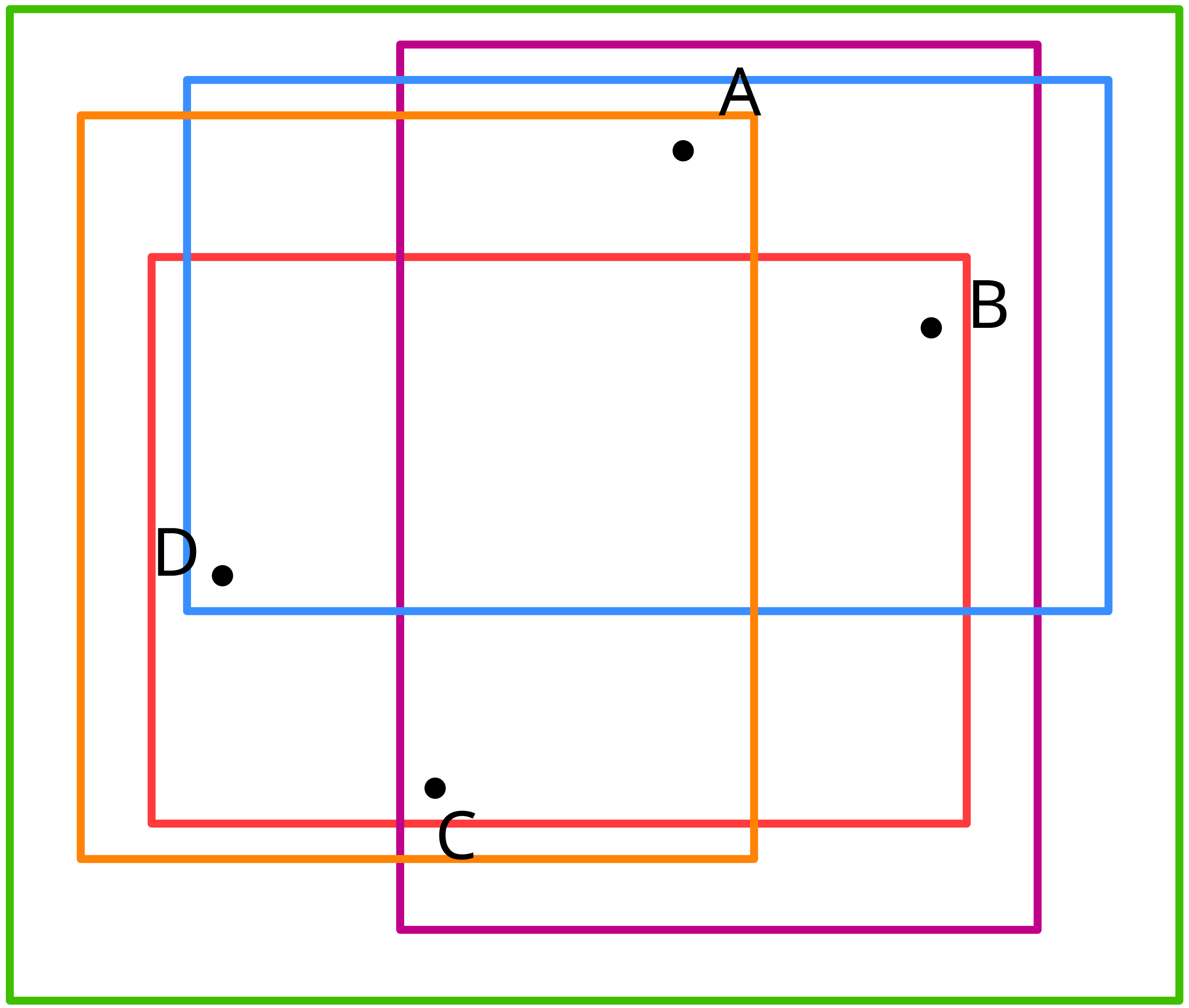


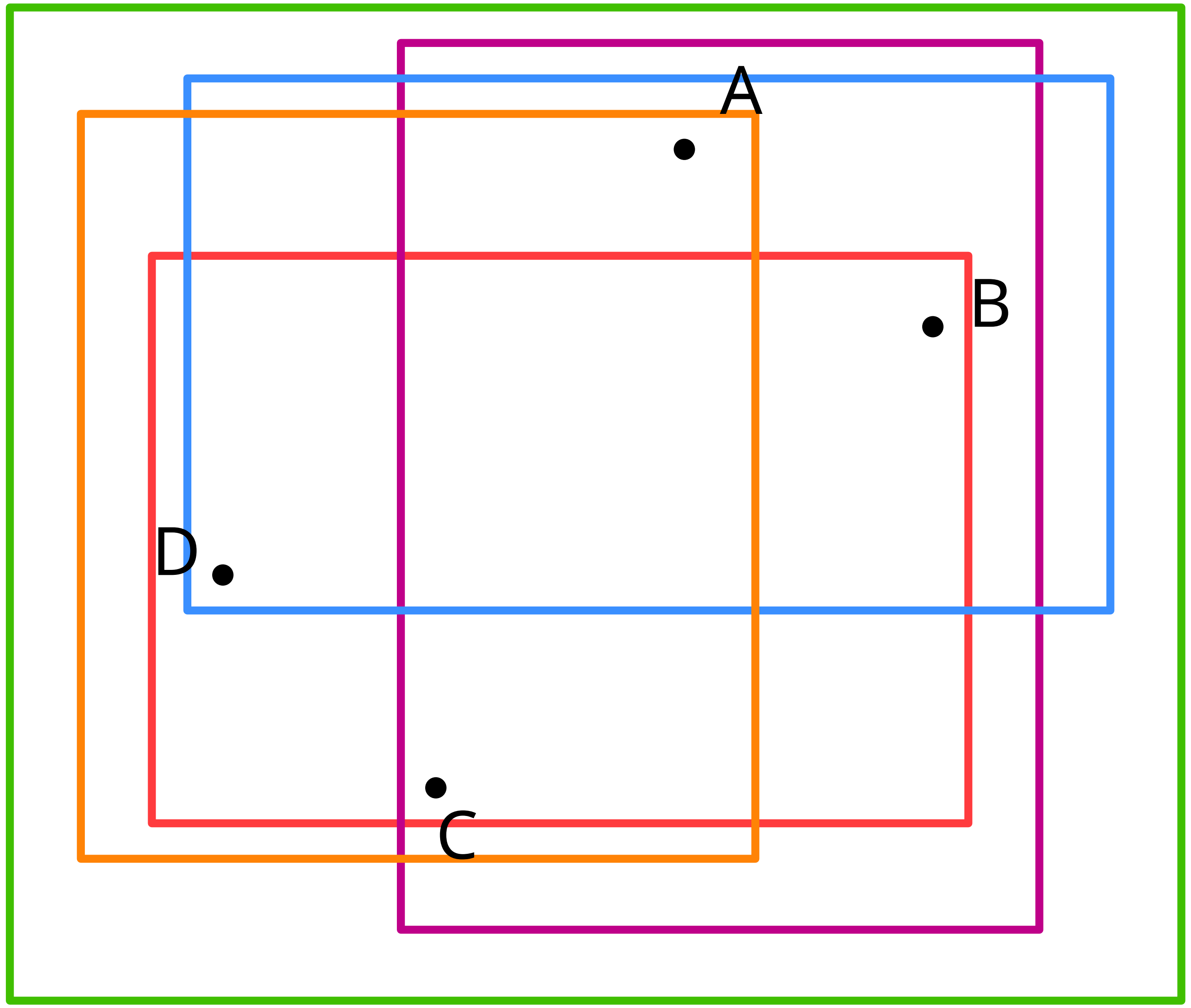
Example: rectangles as ranges

range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles



Example: rectangles as ranges

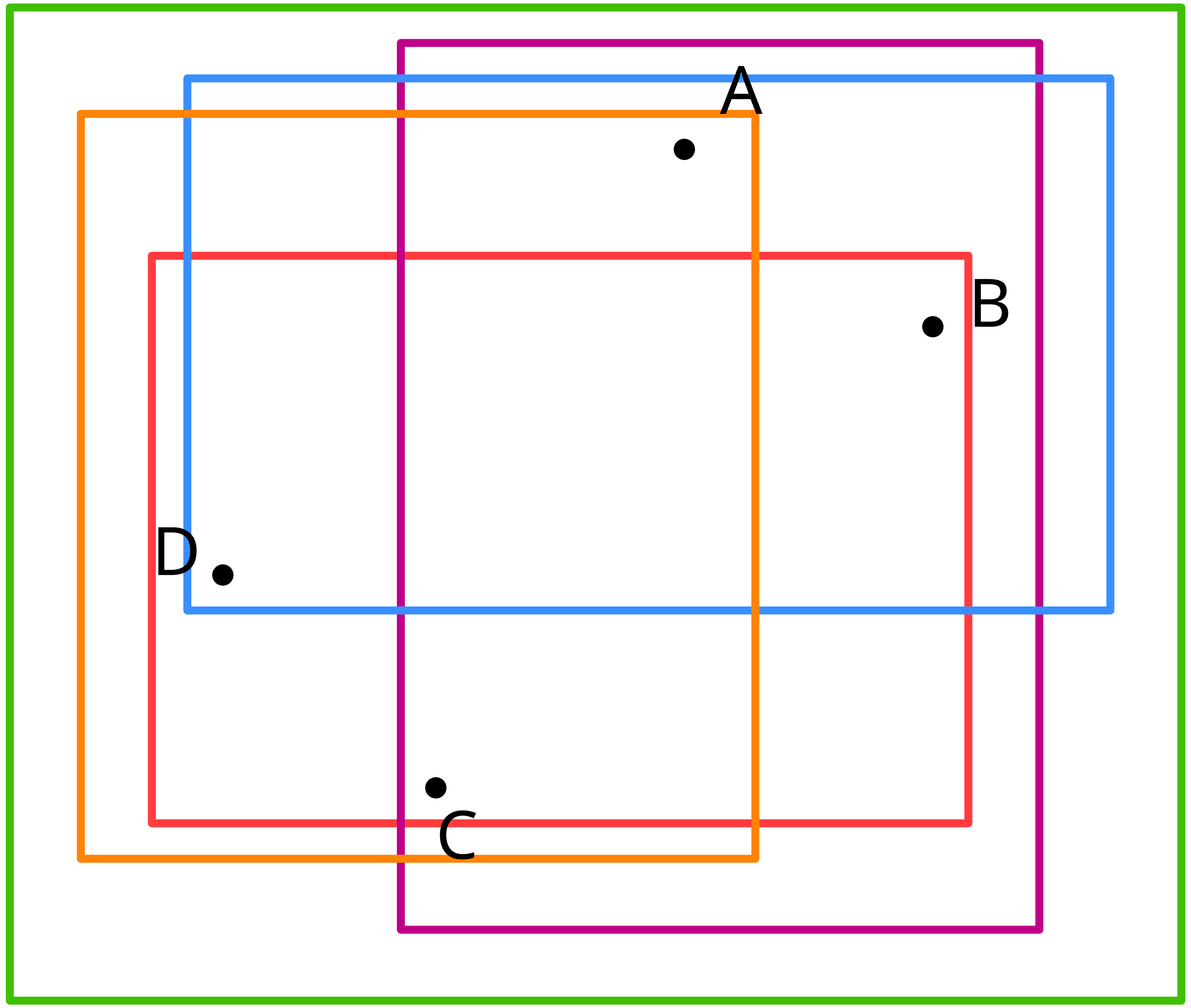
range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles



shattered !

Example: rectangles as ranges

range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles

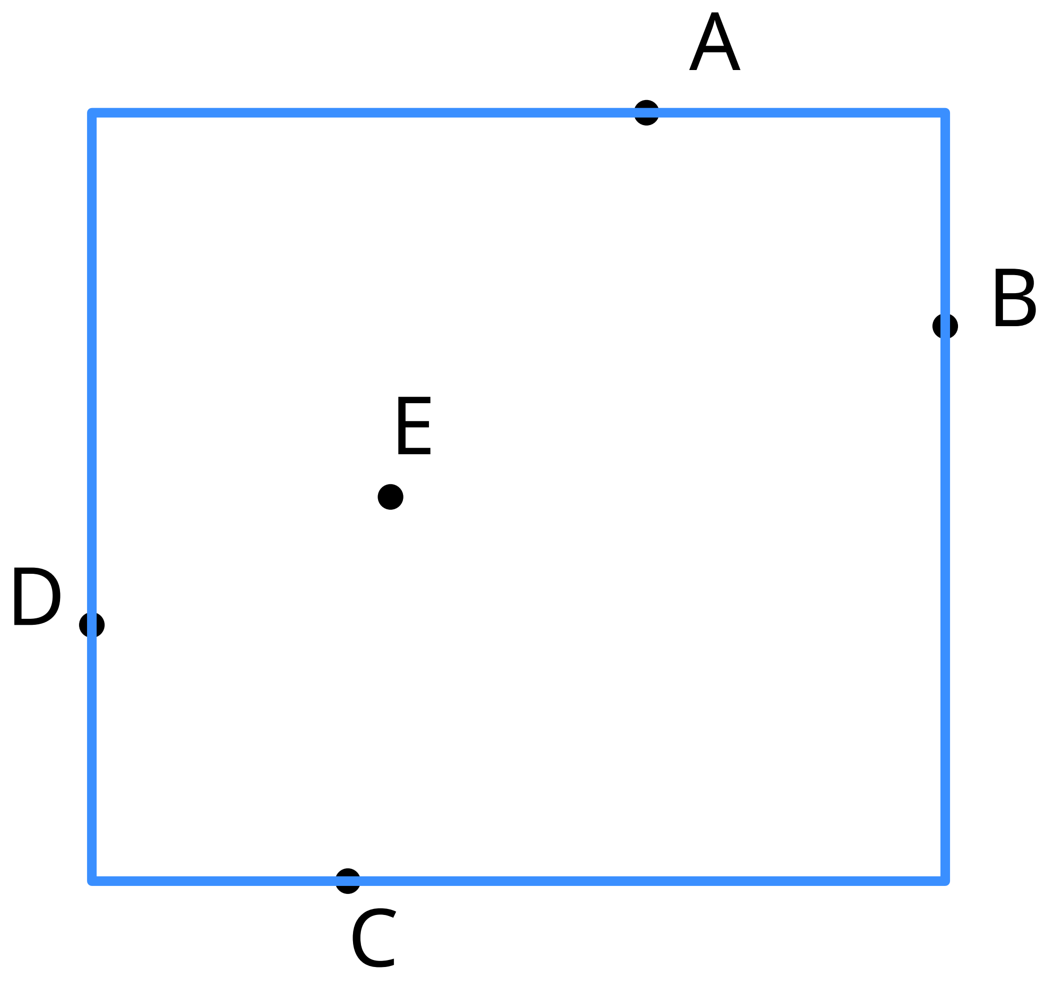


$\Rightarrow$ VC-dimension ≥ 4

shattered !

Example: rectangles as ranges

range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR} =$ set of axis-aligned rectangles



$\Rightarrow$ VC-dimension ≥ 4

Example: rectangles as ranges

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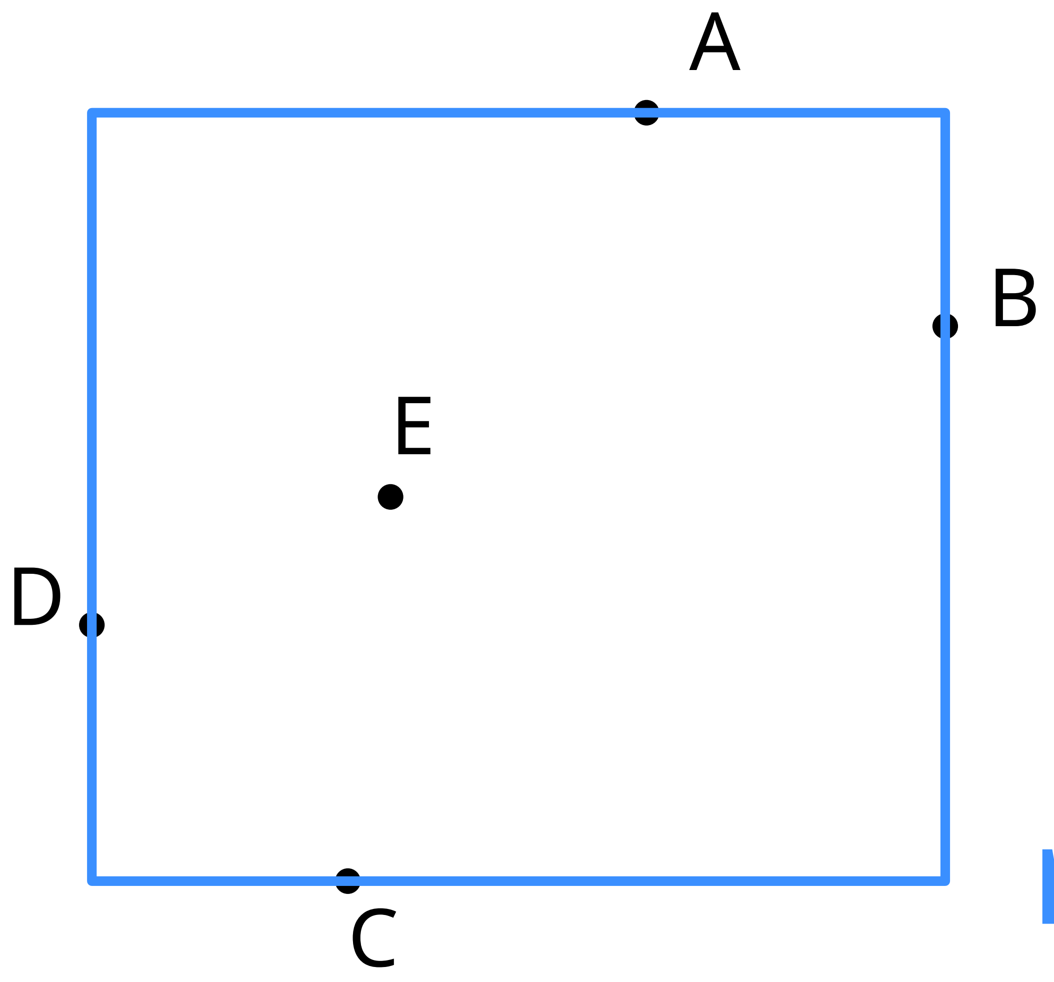
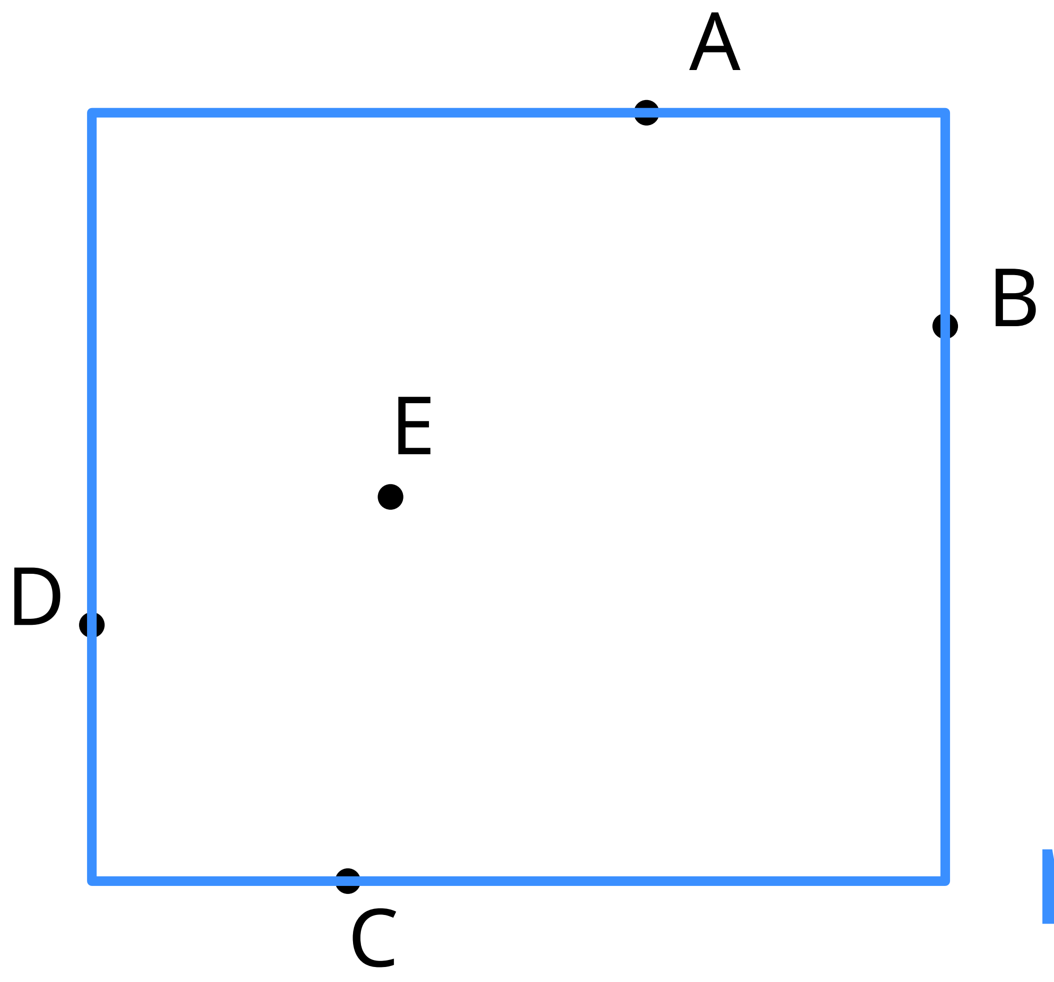


$\Rightarrow$ VC-dimension ≥ 4

not shatter !

Example: rectangles as ranges

range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles



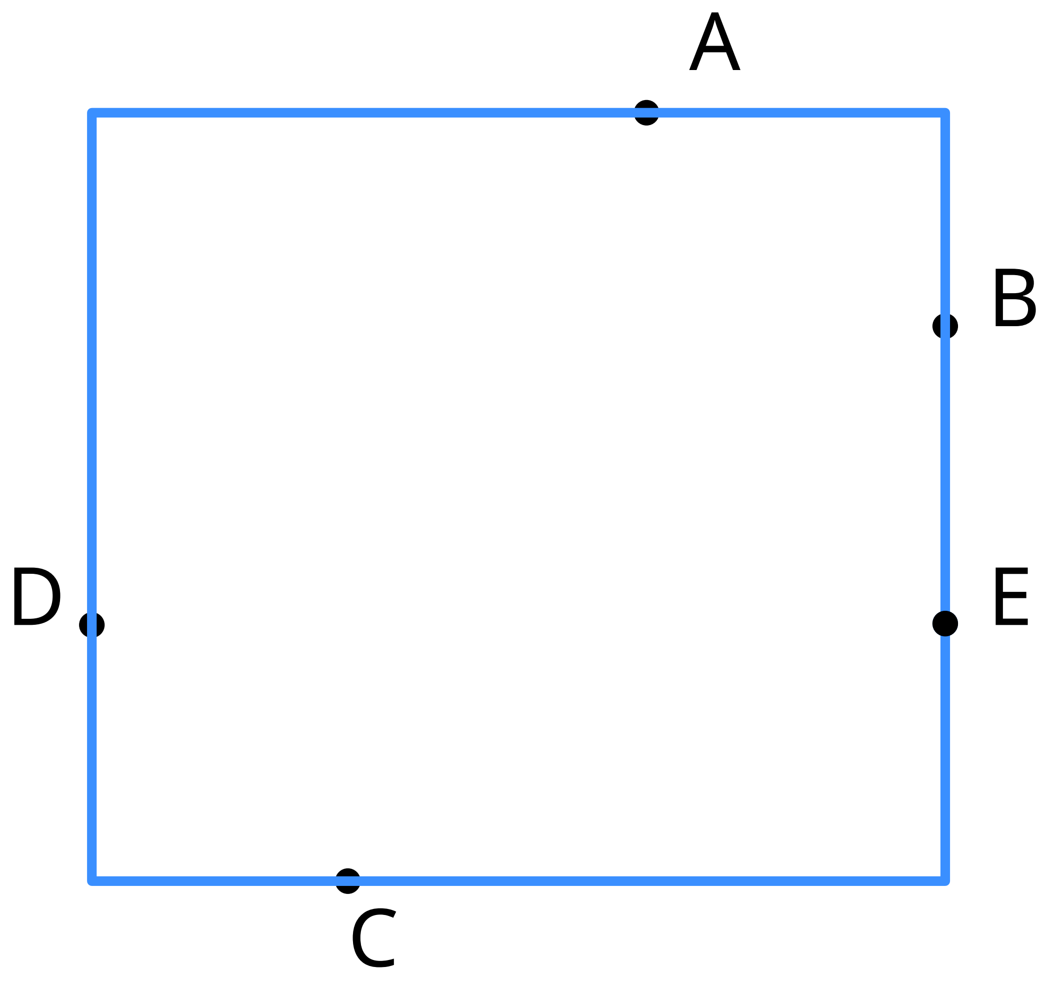
$\Rightarrow$ VC-dimension ≥ 4

case 1: ≥ 1 point inside bounding rectangle

not shatter !

Example: rectangles as ranges

range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR} =$ set of axis-aligned rectangles



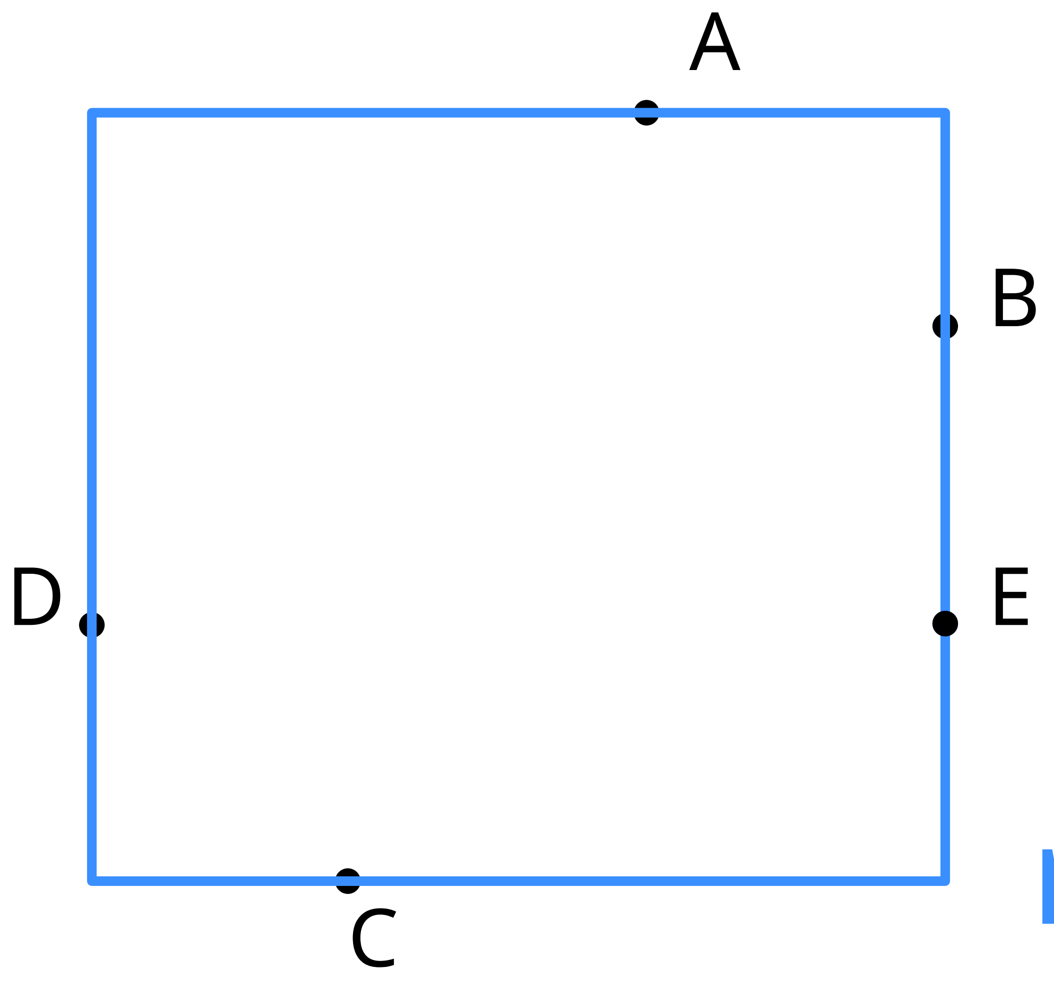
$\Rightarrow$ VC-dimension ≥ 4

case 1: ≥ 1 point inside bounding rectangle

case 2: all points on bounding rectangle

Example: rectangles as ranges

range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR} =$ set of axis-aligned rectangles



$\Rightarrow$ VC-dimension ≥ 4

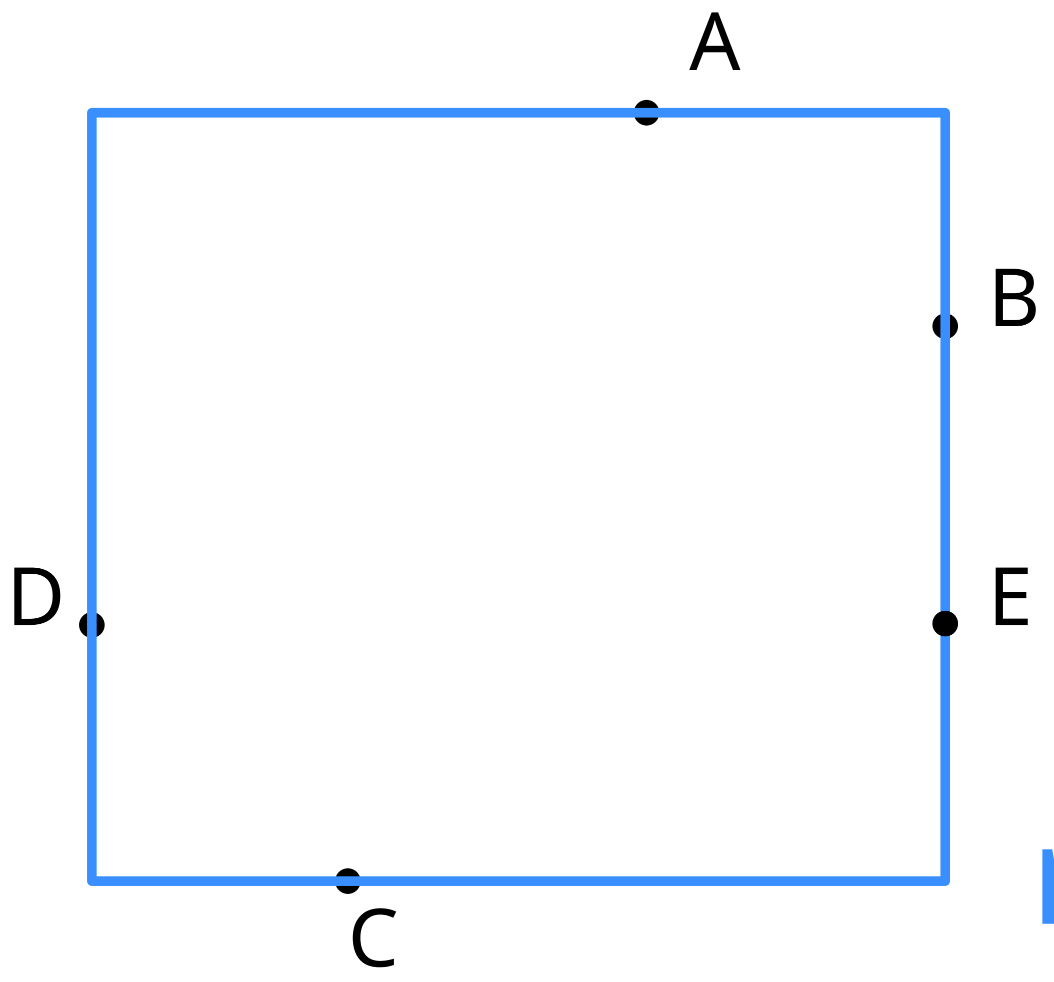
case 1: ≥ 1 point inside bounding rectangle

case 2: all points on bounding rectangle

not shatter !

Example: rectangles as ranges

range space $(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR} =$ set of axis-aligned rectangles



$\Rightarrow$ VC-dimension ≥ 4

case 1: ≥ 1 point inside bounding rectangle

case 2: all points on bounding rectangle

$\Rightarrow$ VC-dimension $= 4$

not shatter !

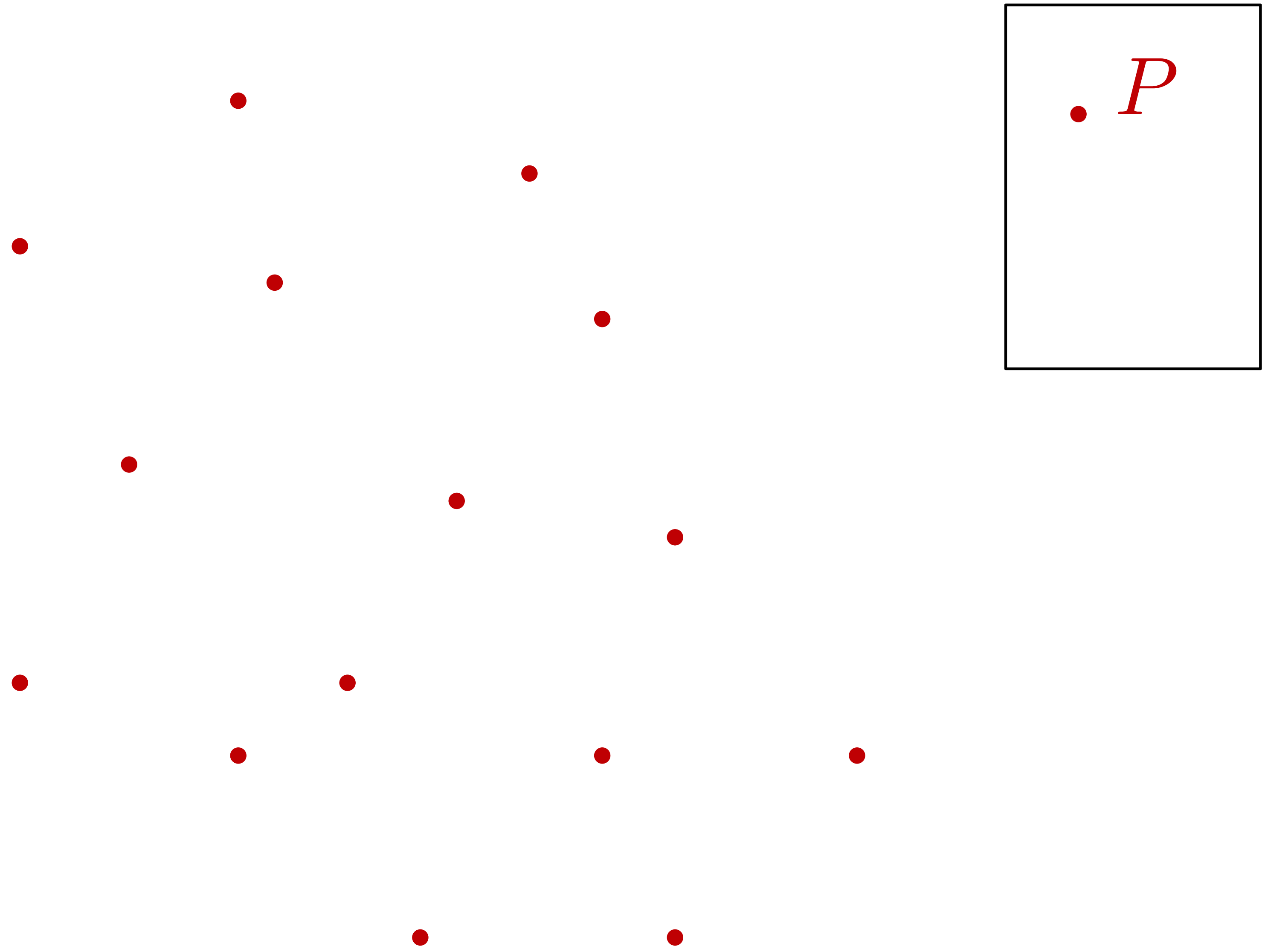
Summary: VC-dimension of geometric range spaces

range space	VC-dimension
$(\mathbb{R}, \mathcal{I})$, with $\mathcal{I} =$ set of closed intervals	2
$(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D} =$ set of disks	3
$(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR} =$ set of axis-aligned rectangles	4
$(\mathbb{R}^2, \mathcal{GR})$, with $\mathcal{GR} =$ set of arbitrary oriented rectangles	? ≥ 4
$(\mathbb{R}^2, \mathcal{C})$, with $\mathcal{C} =$ set of closed convex sets	∞

ε-samples

Measure and Estimate

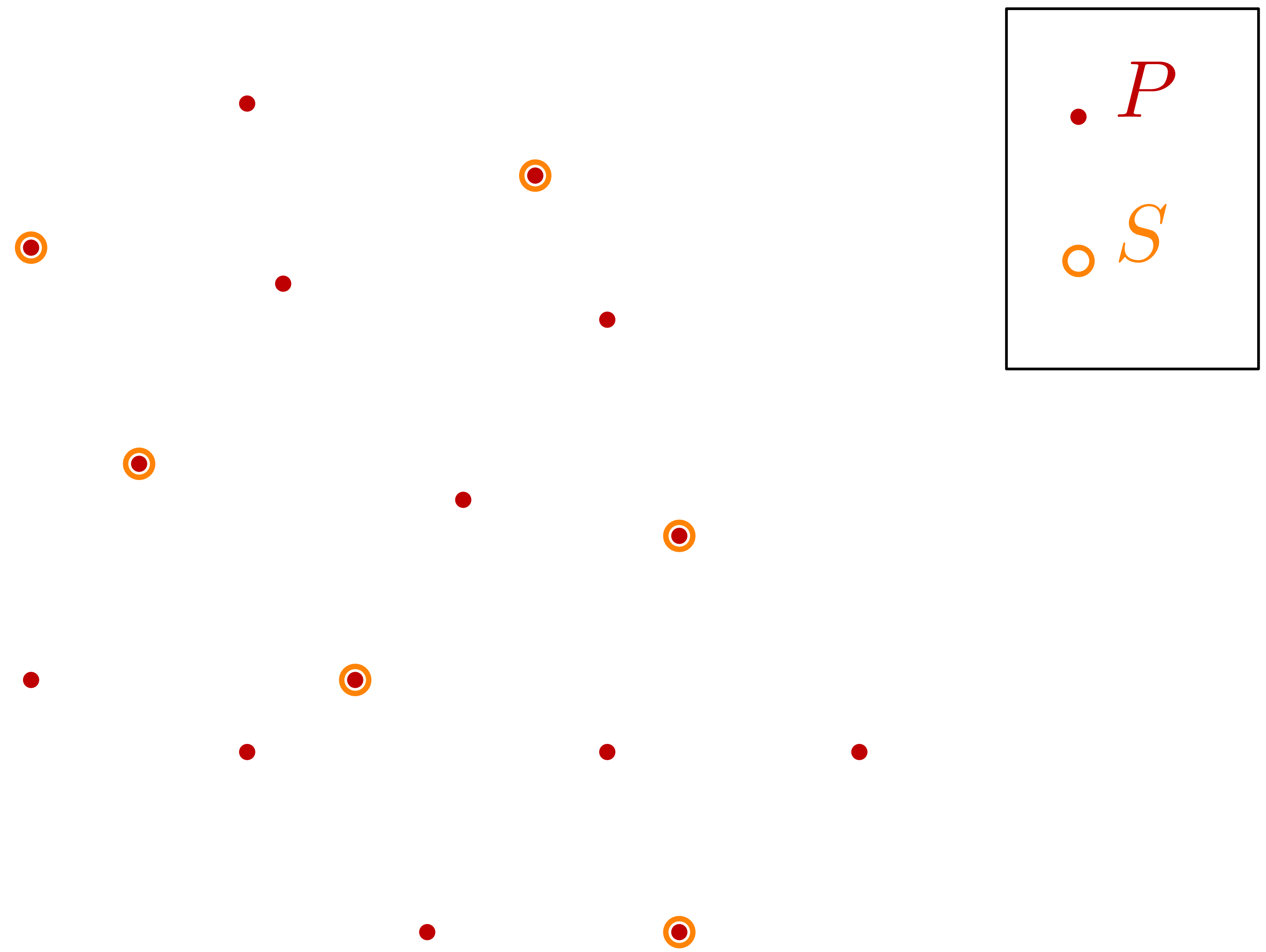
Measure: $\mu(r) = \frac{|r \cap P|}{|P|}$



Measure and Estimate

Measure: $\mu(r) = \frac{|r \cap P|}{|P|}$

Estimate: $\hat{\mu}(r) = \frac{|r \cap S|}{|S|}$



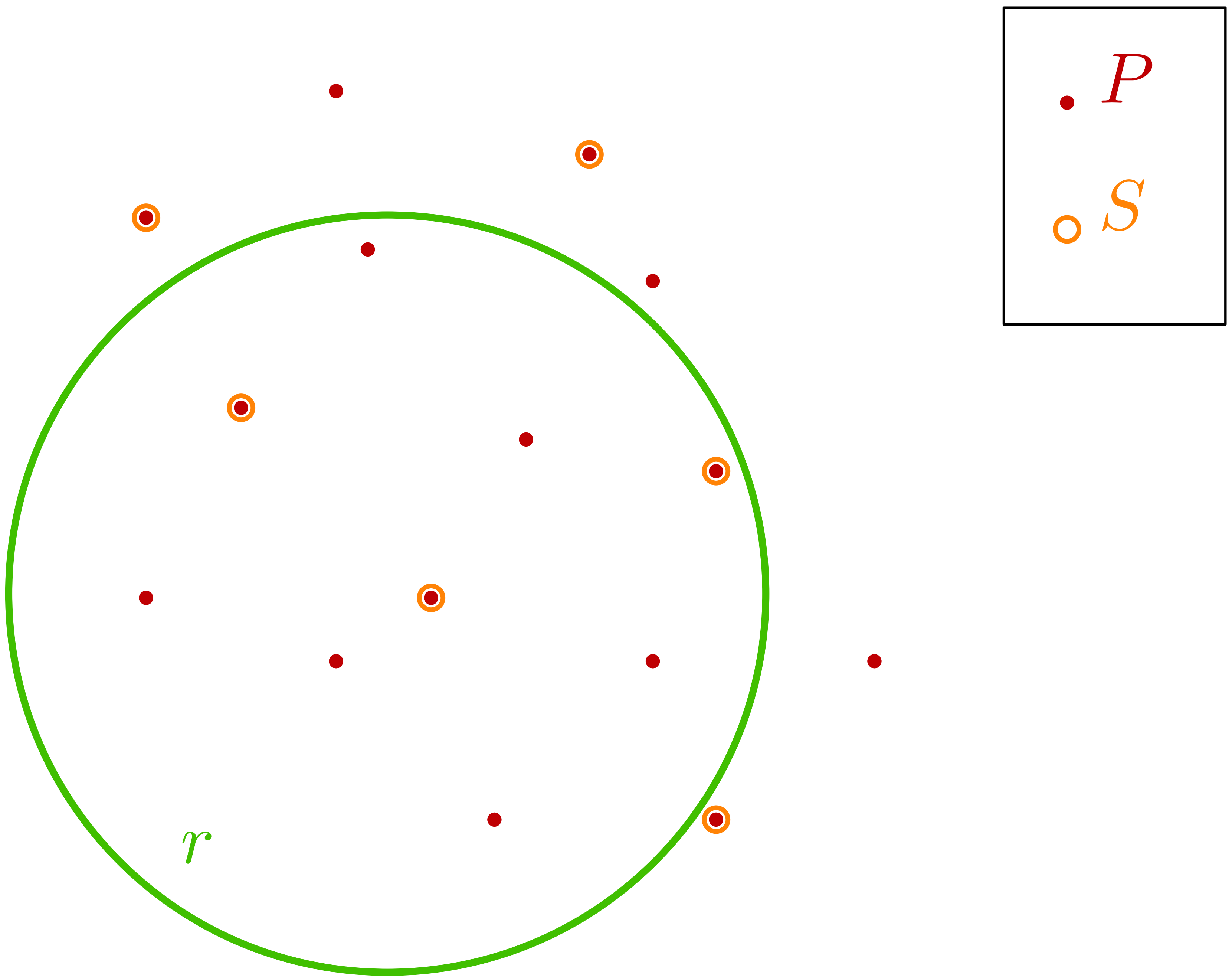
Measure and Estimate

Measure: $\mu(r) = \frac{|r \cap P|}{|P|}$

$\mu(Q) = \frac{9}{15} = 0.6$

Estimate: $\hat{\mu}(r) = \frac{|r \cap S|}{|S|}$

$\hat{\mu}(Q) = \frac{3}{6} = 0.5$



Measure and Estimate

Measure: $\mu(r) = \frac{|r \cap P|}{|P|}$

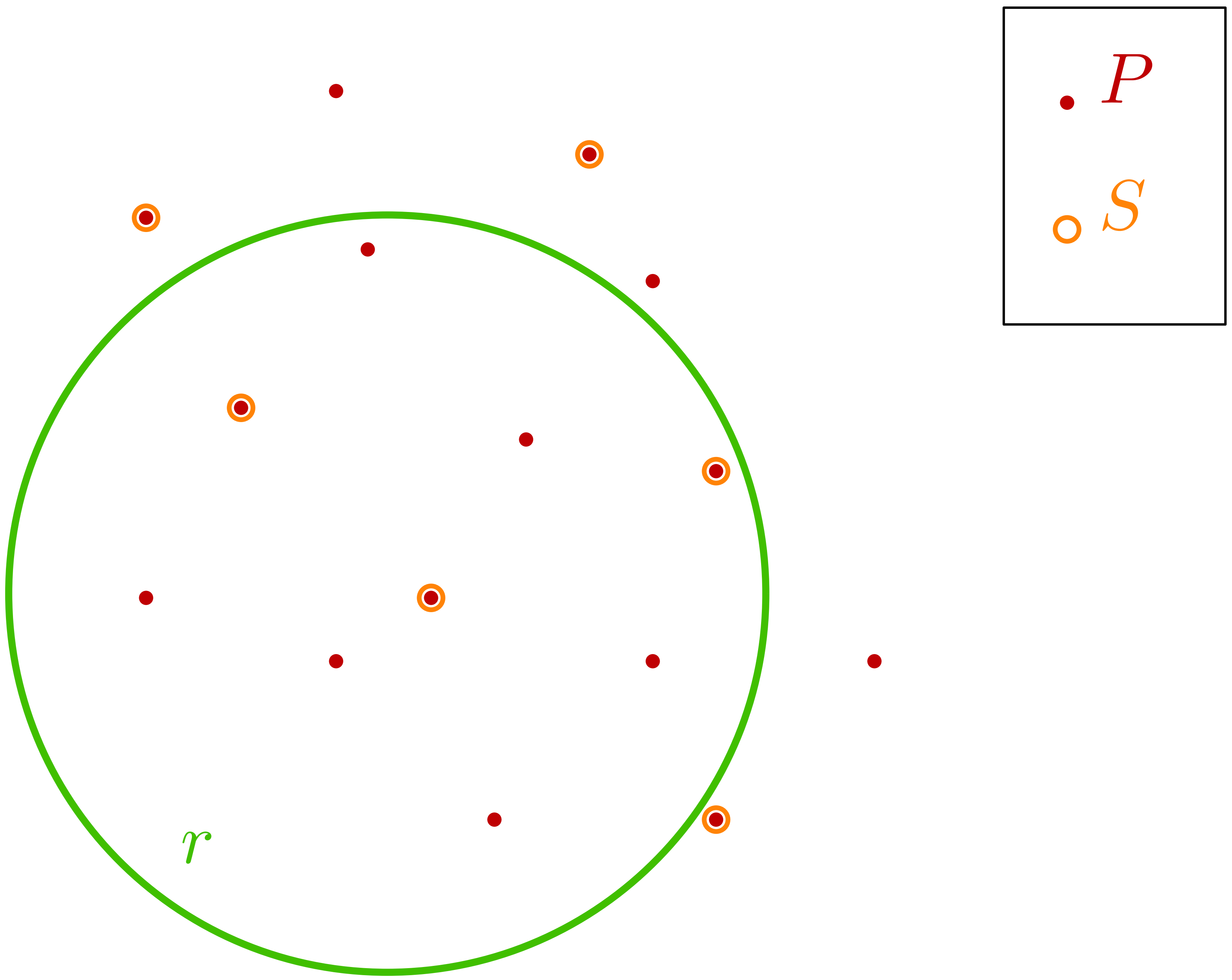
$\mu(Q) = \frac{9}{15} = 0.6$

Estimate: $\hat{\mu}(r) = \frac{|r \cap S|}{|S|}$

$\hat{\mu}(Q) = \frac{3}{6} = 0.5$

Good Sample S:

for all $r \in \mathcal{R}$, $\hat{\mu}(r) \approx \mu(r)$



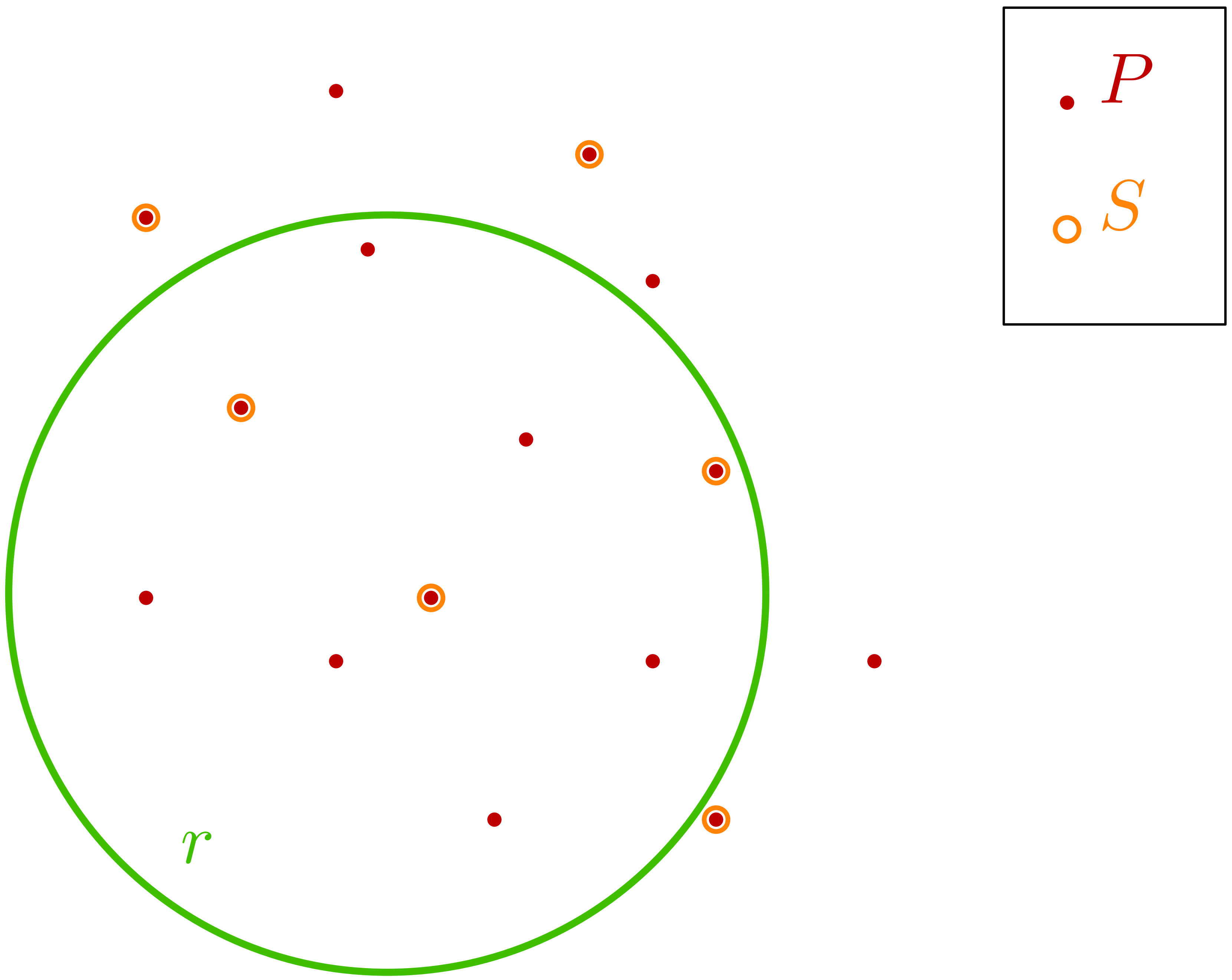
ε-samples

ε-sample S:

for all $r \in \mathcal{R}$ and any

$0 \leq \varepsilon \leq 1$

$|\mu(r) - \hat{\mu}(r)| \leq \varepsilon$



ε-samples

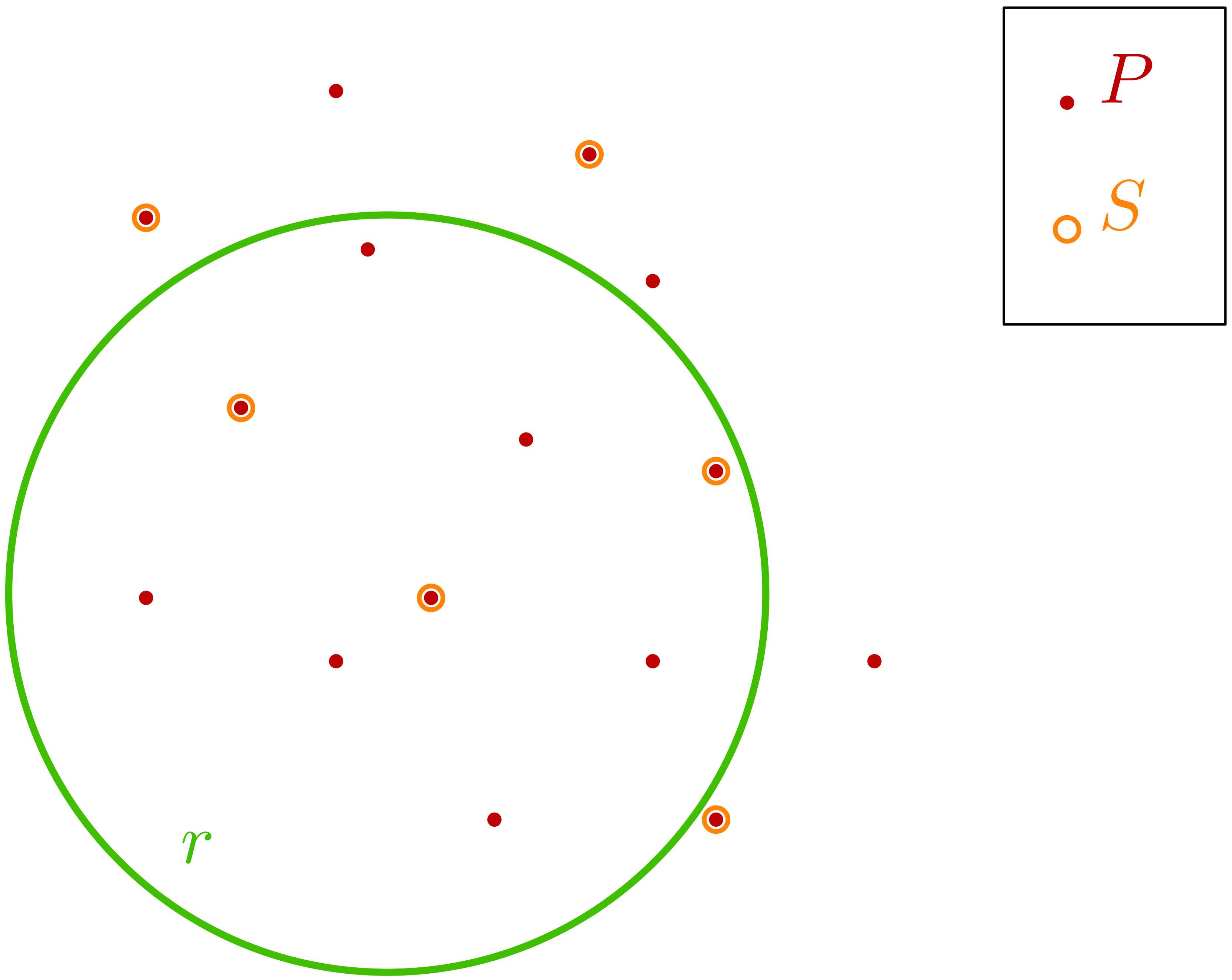
ε-sample S:

for all $r \in \mathcal{R}$ and any

$0 \leq \varepsilon \leq 1$

$|\mu(r) - \hat{\mu}(r)| \leq \varepsilon$

$$|\mu(r) - \hat{\mu}(r)| = |9/15 - 3/6| = 0.1$$



Quiz

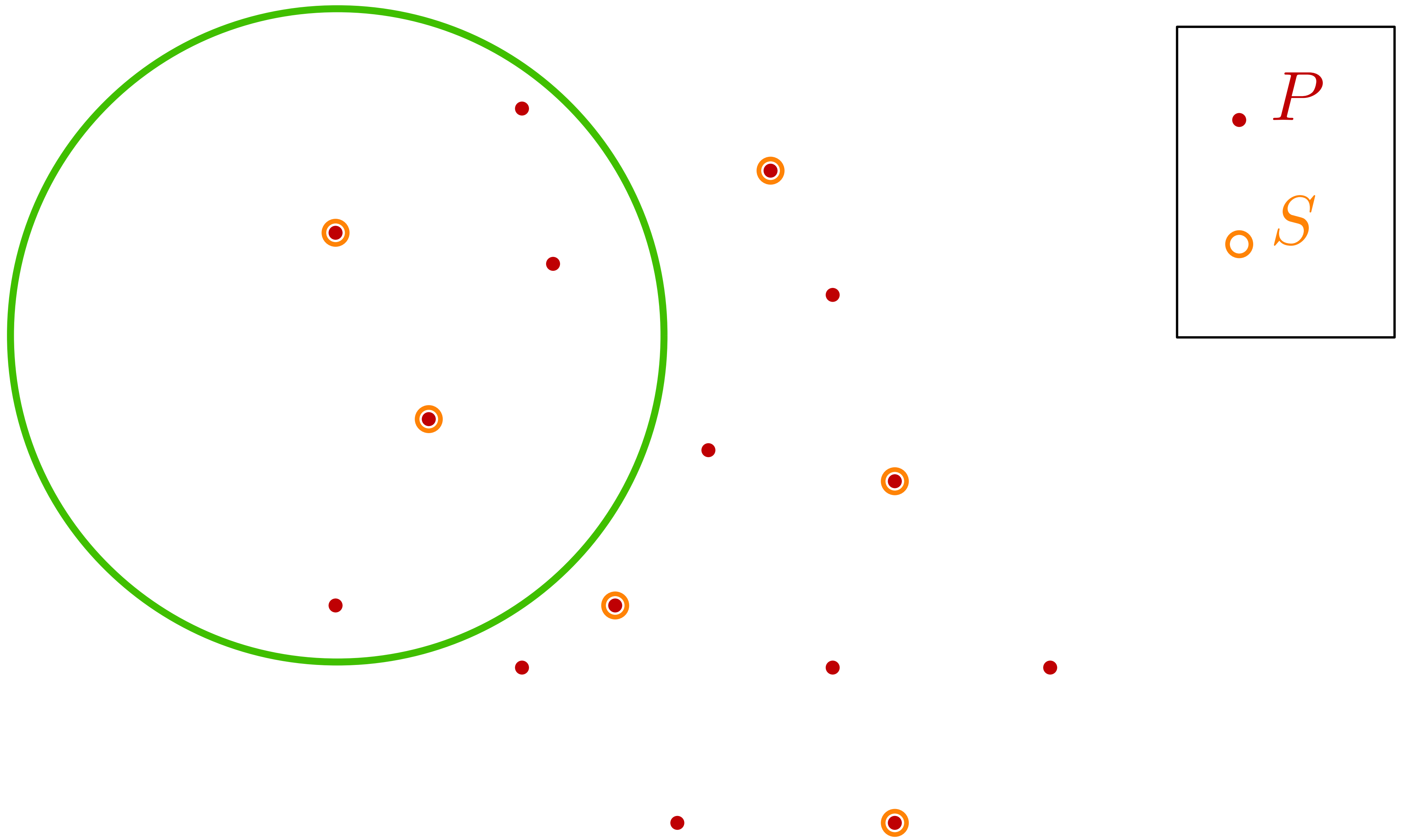
$|\mu(r) - \hat{\mu}(r)| = \ldots ?$

A 0.0

B 0.1

C 0.2

D none of the above



Quiz

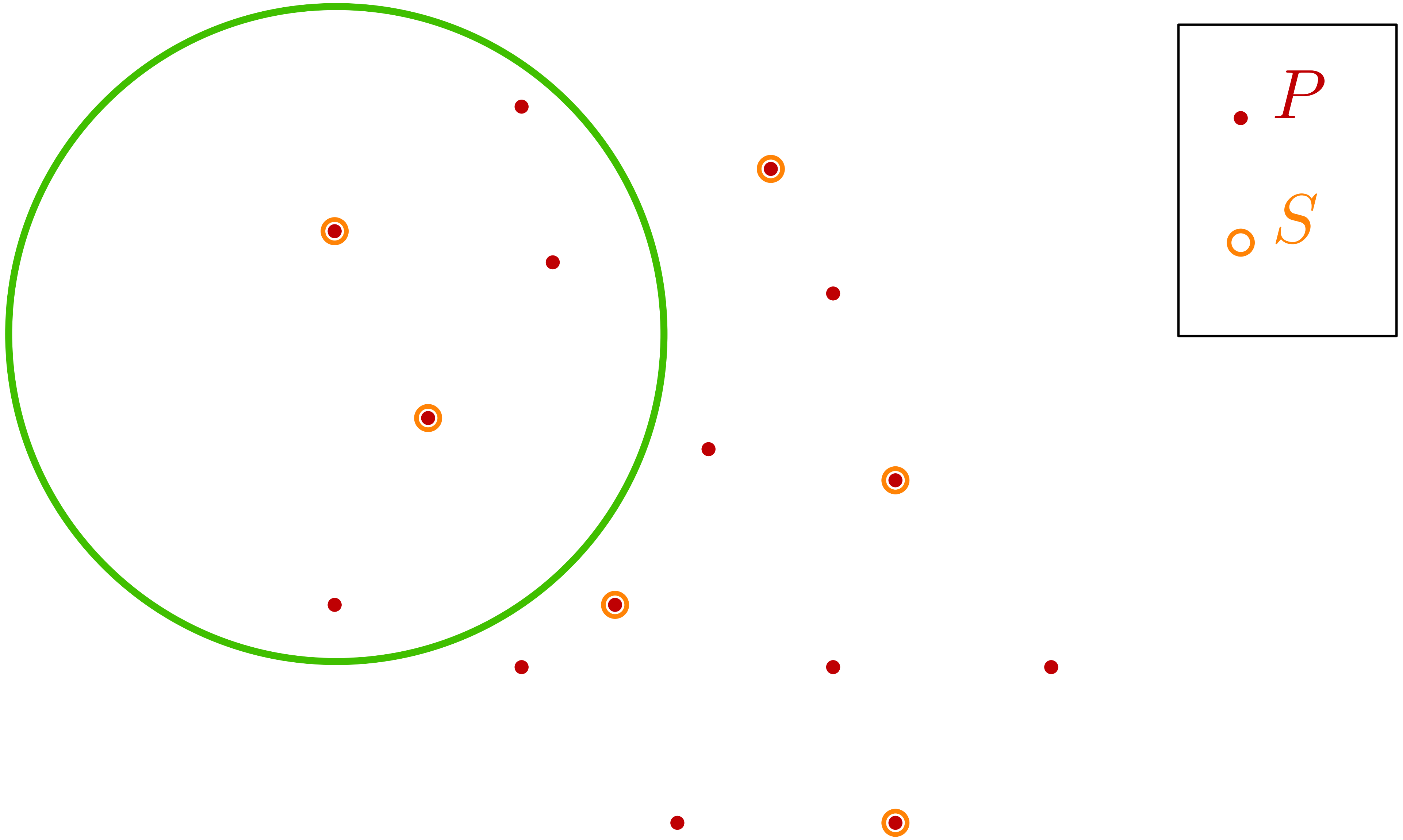
$$|\mu(r) - \hat{\mu}(r)| = \ldots ?$$

A 0.0 $\frac{2}{6} = \frac{5}{15}$

B 0.1

C 0.2

D none of the above



ε-sample theorem

Let $\varphi, \varepsilon > 0$ be parameters and $(X, \mathcal{R})$ be a range space with finite X and VC-dimension δ. A sample of size

$$O\left(\frac{1}{\varepsilon^2}\left(\delta + \log \varphi^{-}1\right)\right)$$

is an ε-sample for $(X, \mathcal{R})$ with probability $\geq 1 - \varphi$

(we skip the proof)

Example from motivation

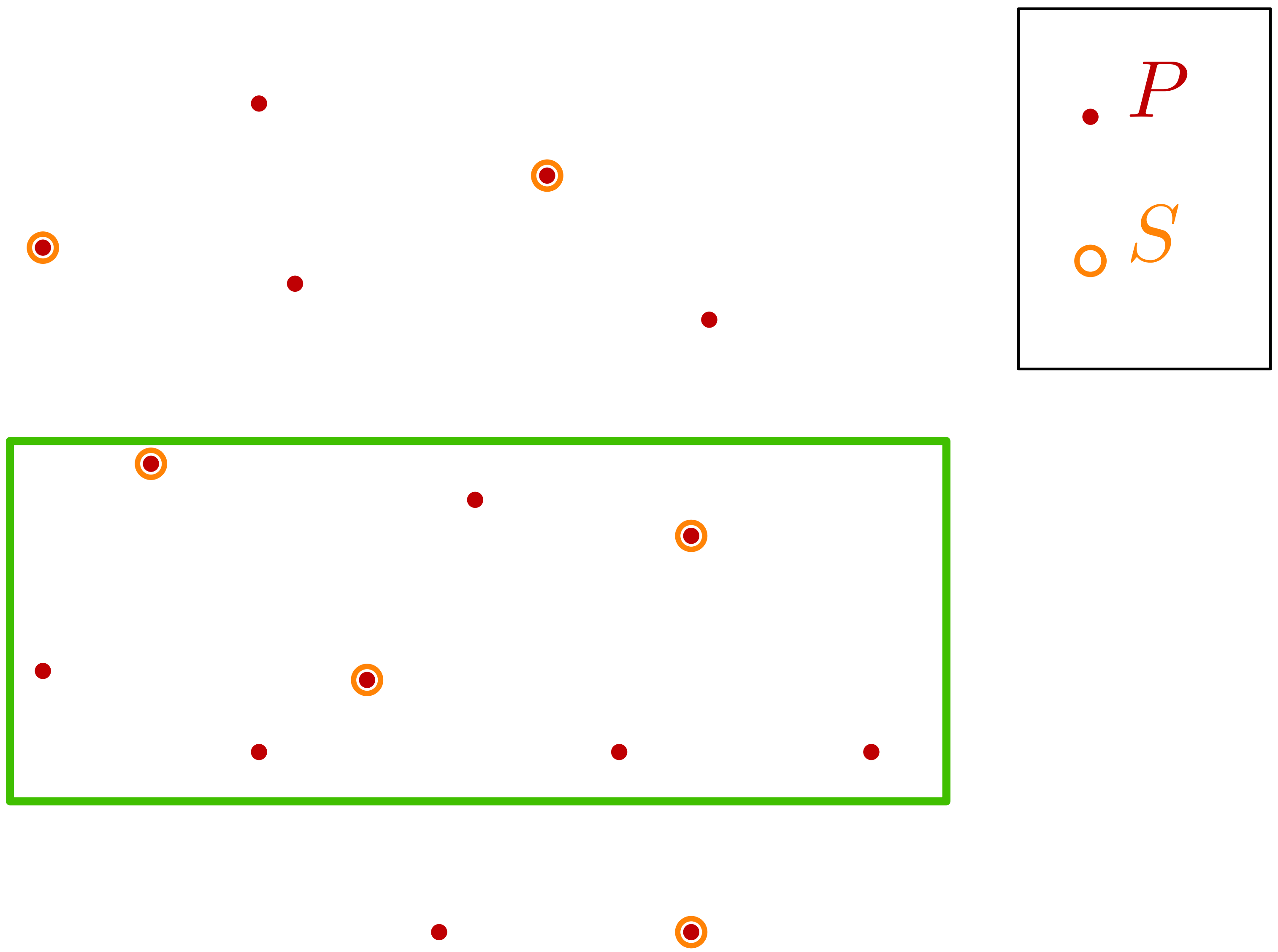
Given P,

how many points do we need to sample ($S \subset P$), such that

2. for any query rectangle r

$$\left| \frac{|r \cap P|}{|P|} - \frac{|r \cap S|}{|S|} \right| \leq 0.25 \qquad ?$$

with probability 0.999



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Given P,

how many points do we need to sample ($S \subset P$), such that

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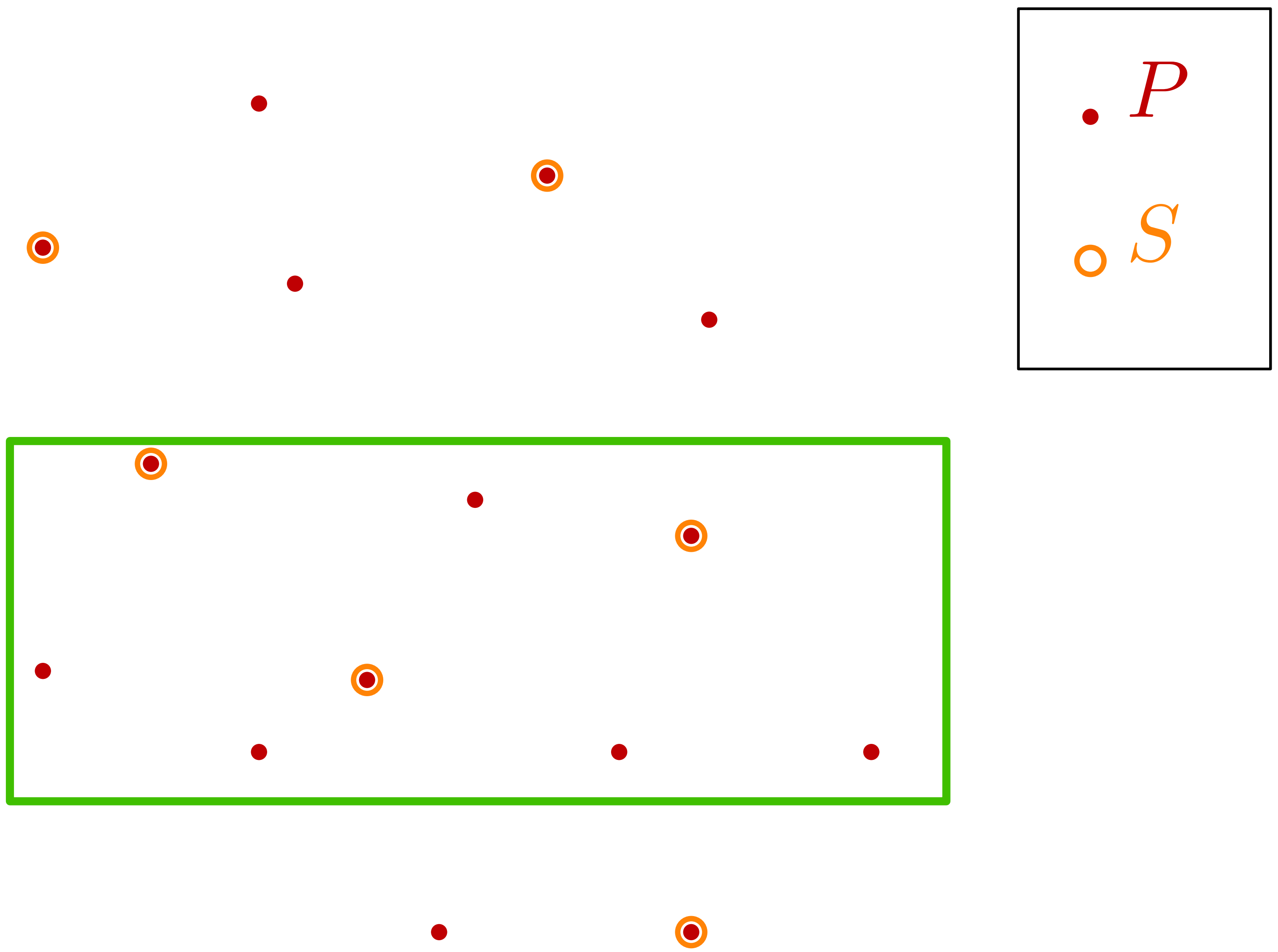

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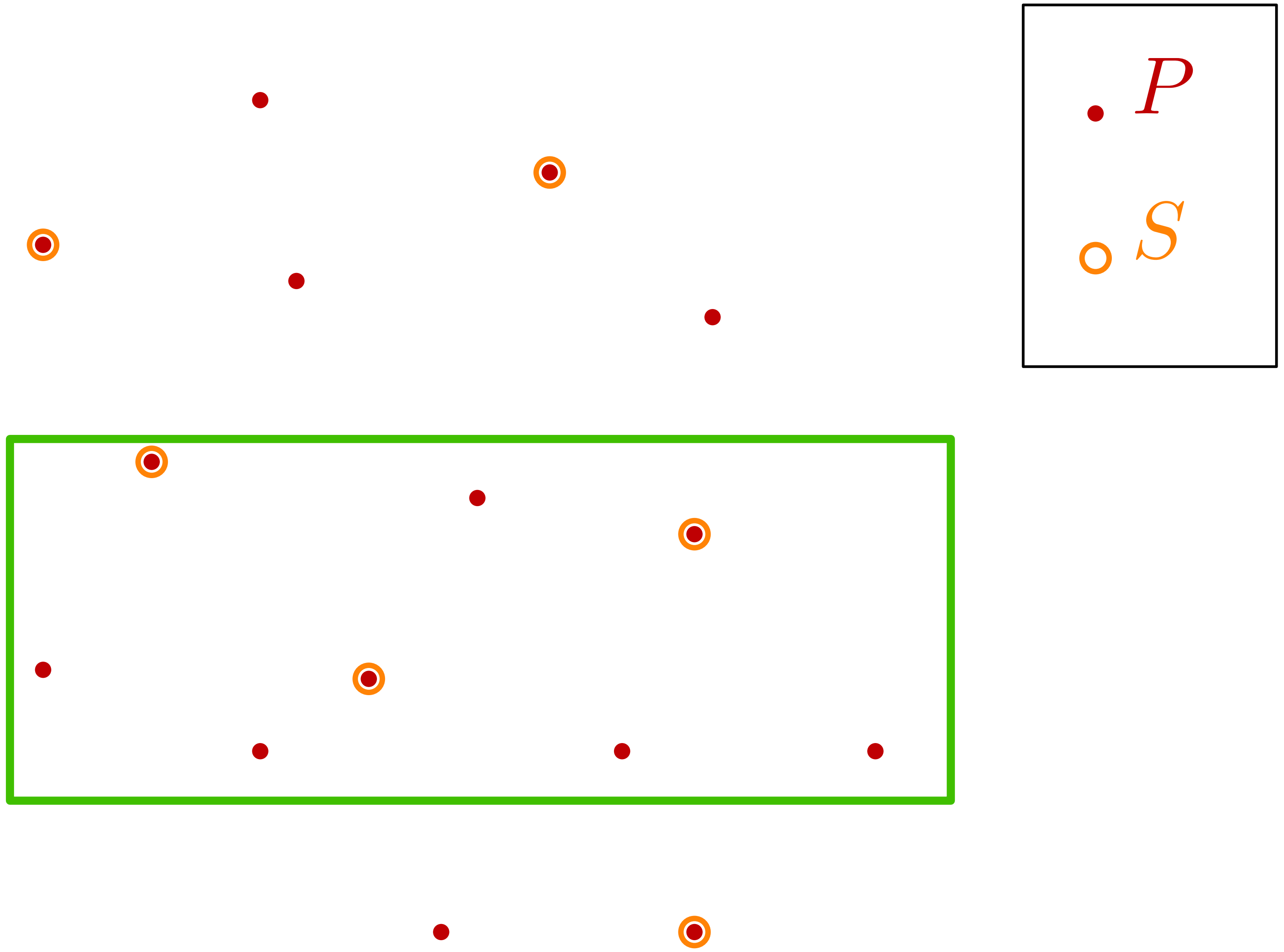

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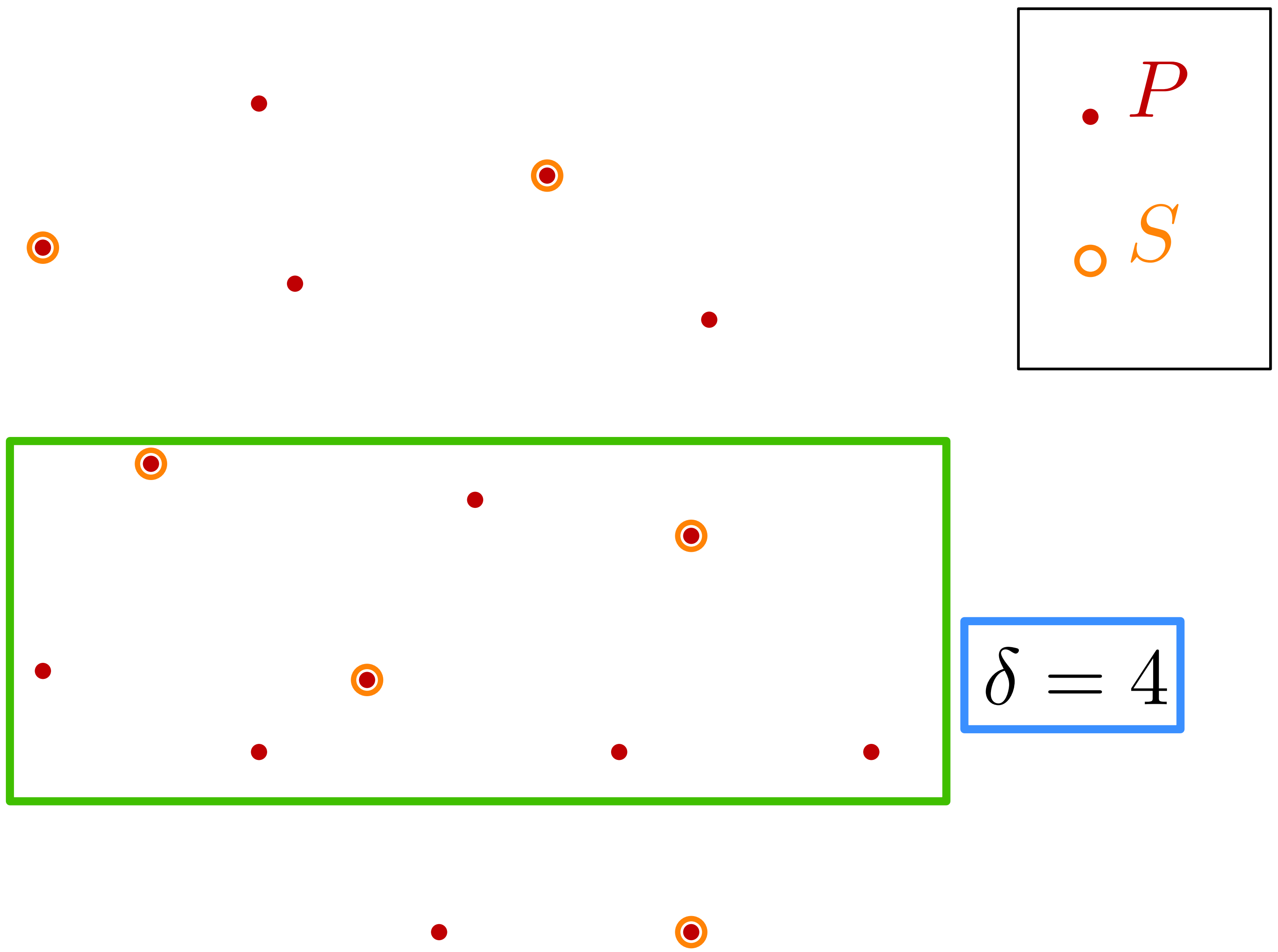

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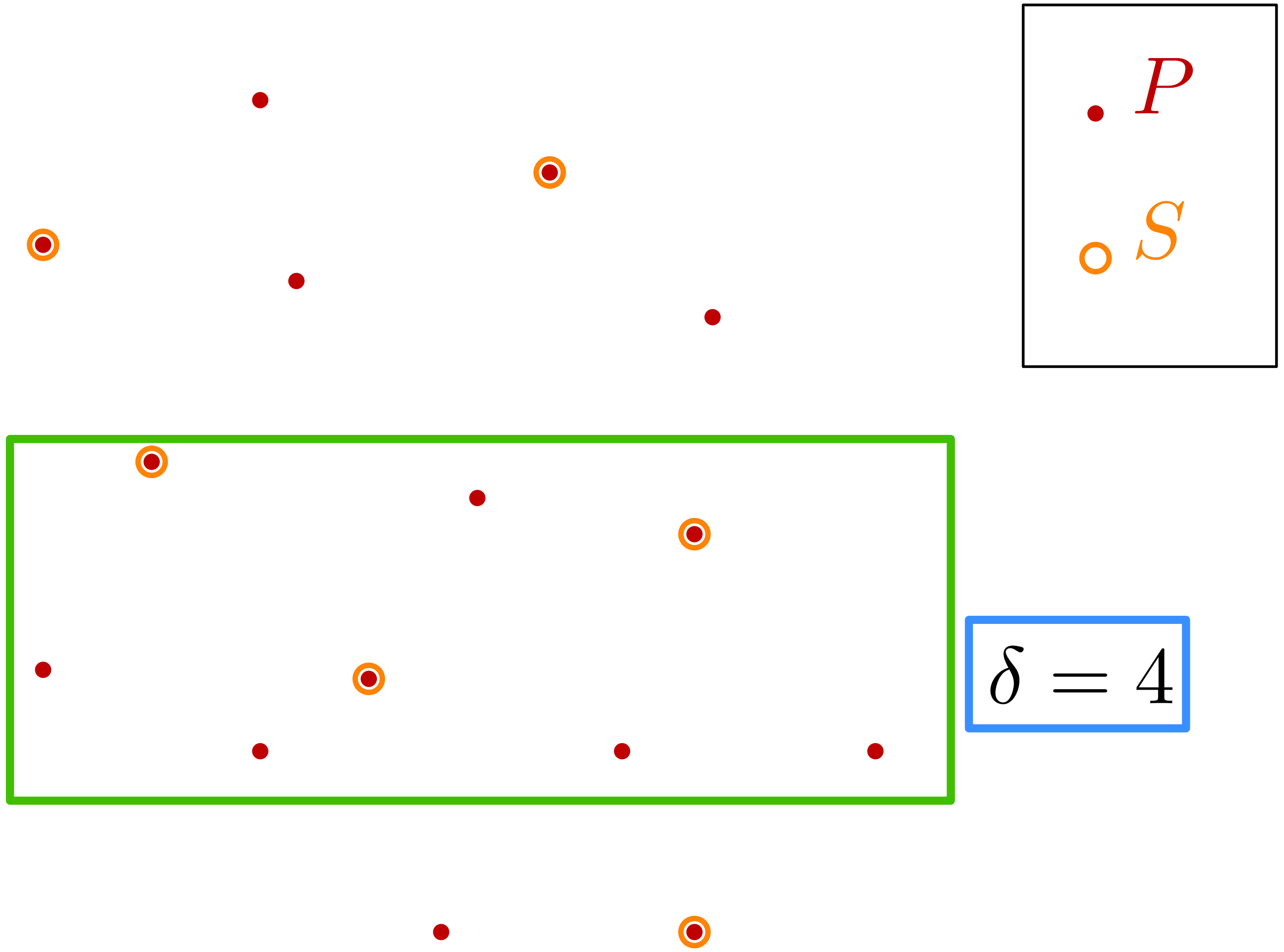

answer: $O\left(\frac{1}{\varepsilon^2}\left(4 + \log \phi^{-}1\right)\right)$,
in particular $O(1)$ for given ε, φ independent of n

ε-nets

ε-nets

ε-sample S:
for all $r \in \mathcal{R}$ and any
$0 \leq \varepsilon \leq 1$
if $\mu(r) \geq \varepsilon$ and
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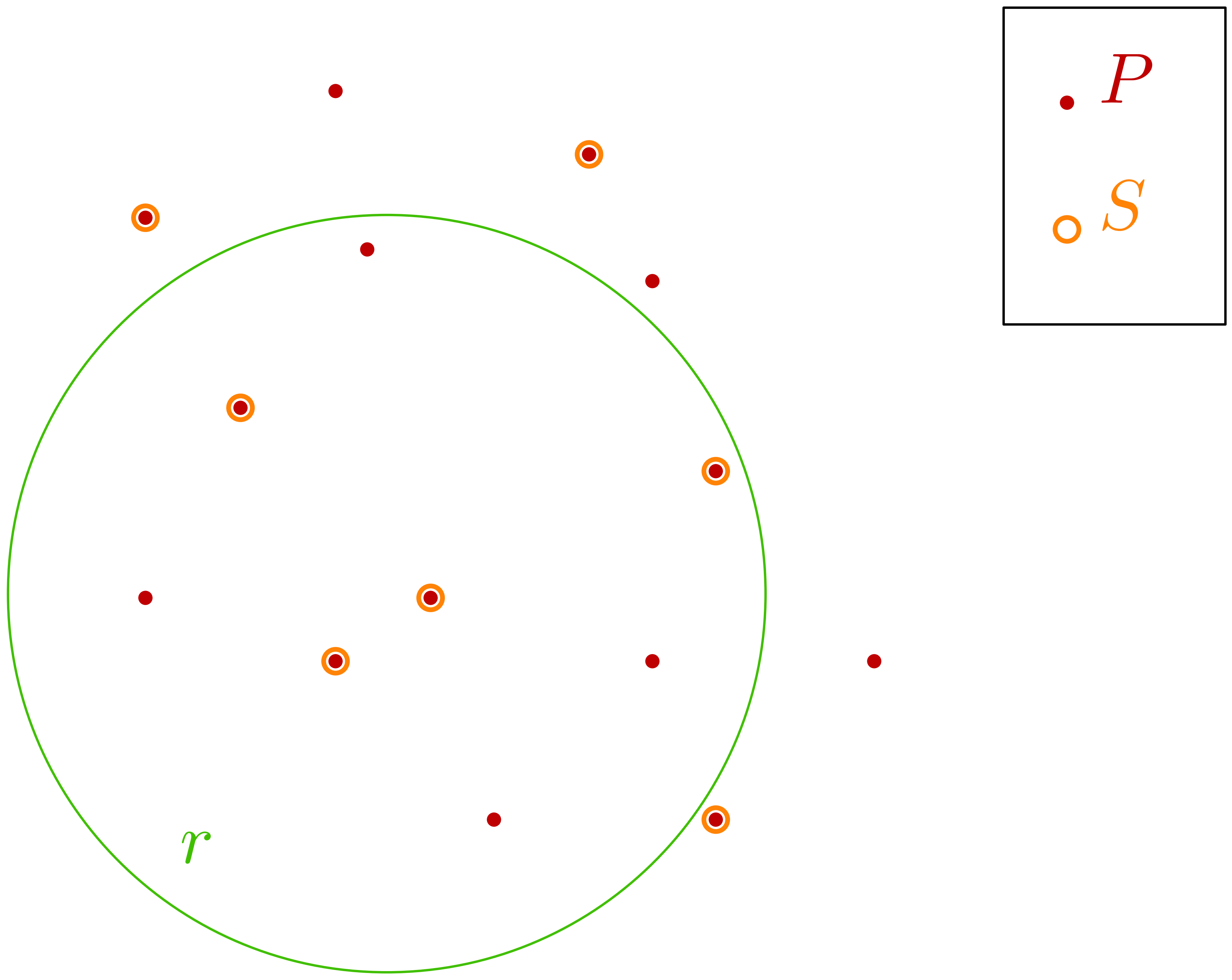

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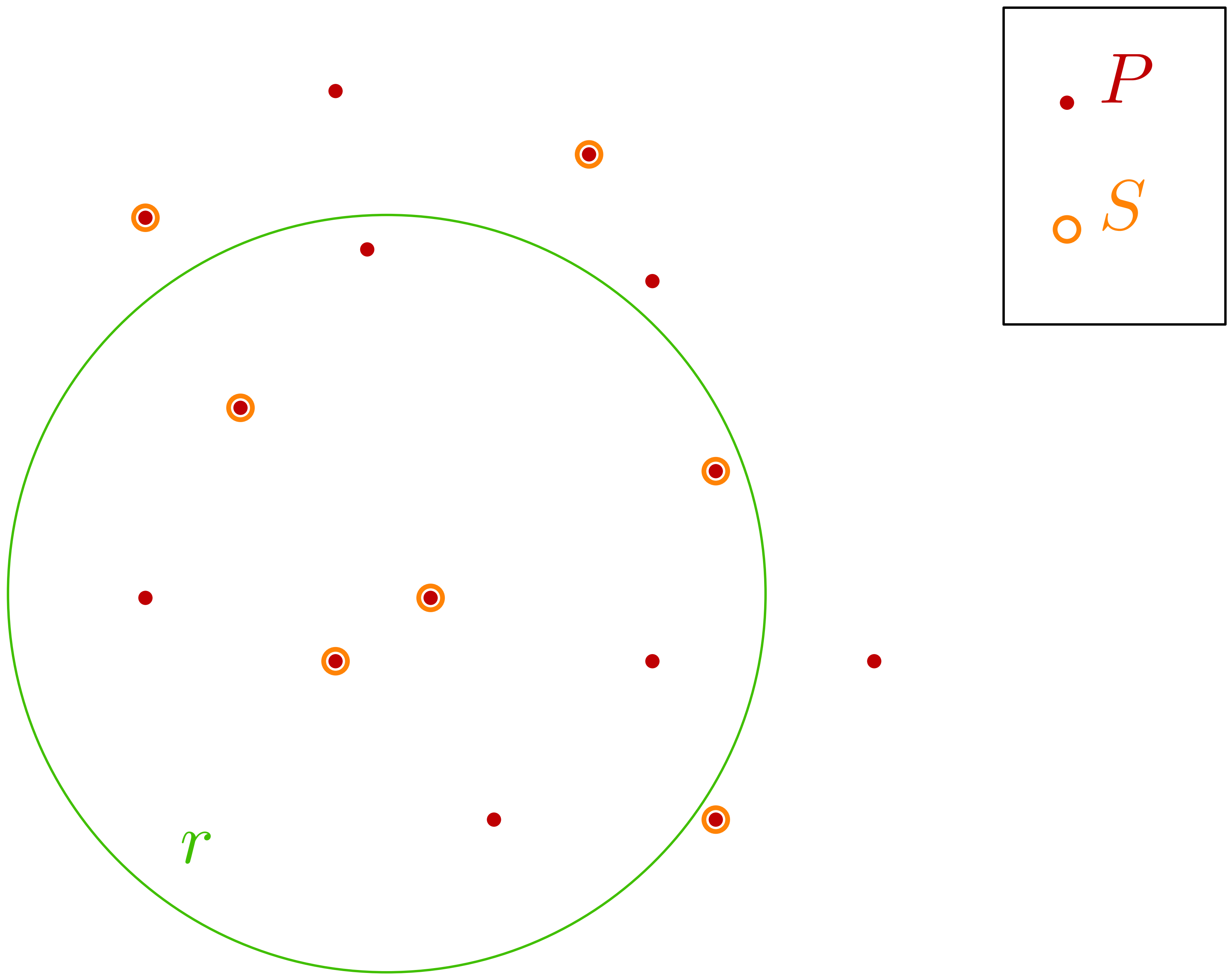

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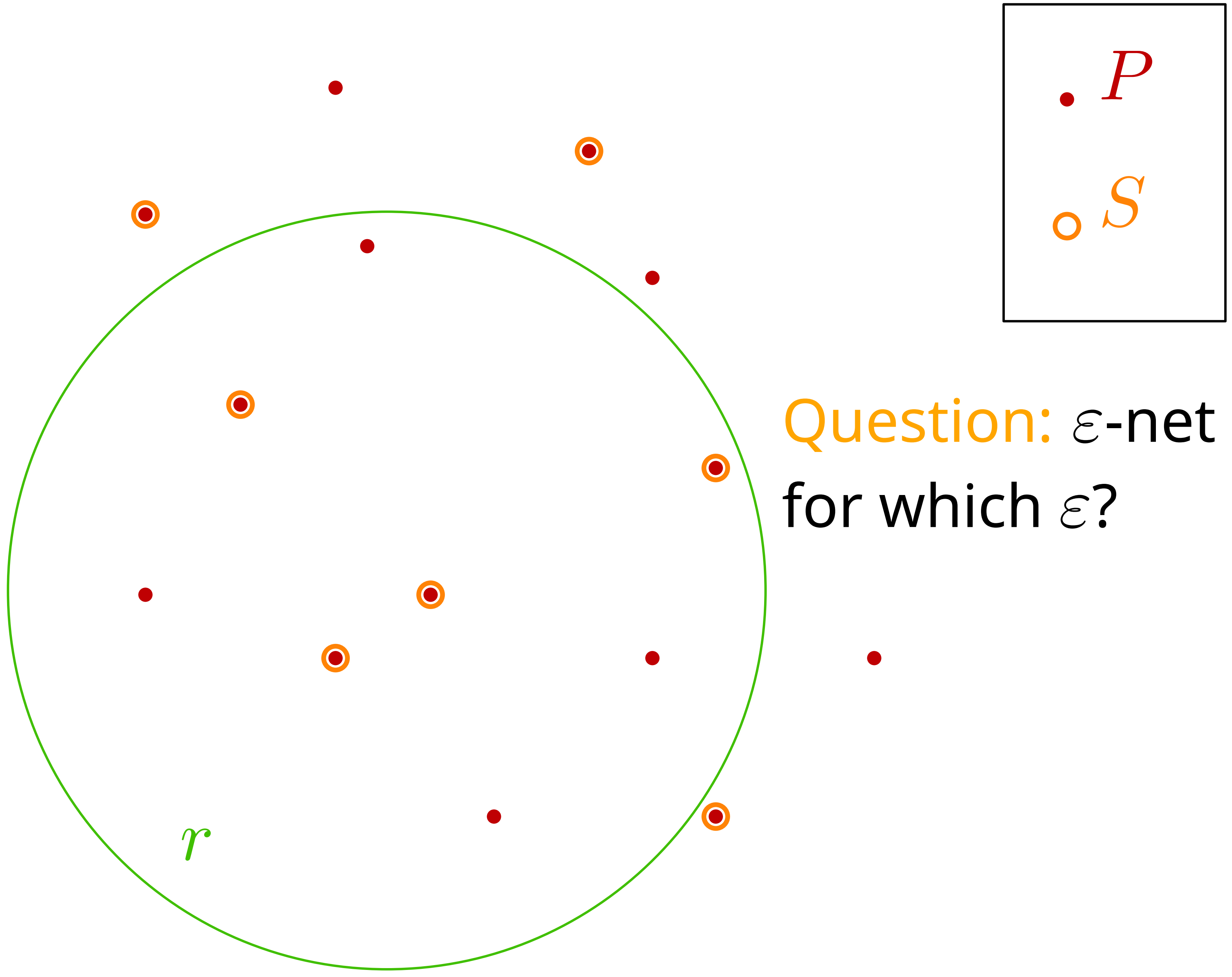

Question: ε-net for which ε?

ε-Net Theorem

Let $\varphi, \varepsilon > 0$ be parameters and $(X, \mathcal{R})$ be a range space with finite X and VC-dimension δ. A sample obtained by m random draws from X with

$$m \geq \max\left(\frac{4}{\varepsilon}\log\frac{4}{\varphi}, \frac{8\delta}{\varepsilon}\log\frac{16}{\varepsilon}\right)$$

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in short: ε-sample $O\left(\frac{\delta}{\varepsilon^2}\right)$ vs ε-net $O\left(\frac{\delta}{\varepsilon}\log\frac{1}{\varepsilon}\right)$

Motivation: sampling for approximation

Given P, how many points do we need to sample ($S \subset P$), such that

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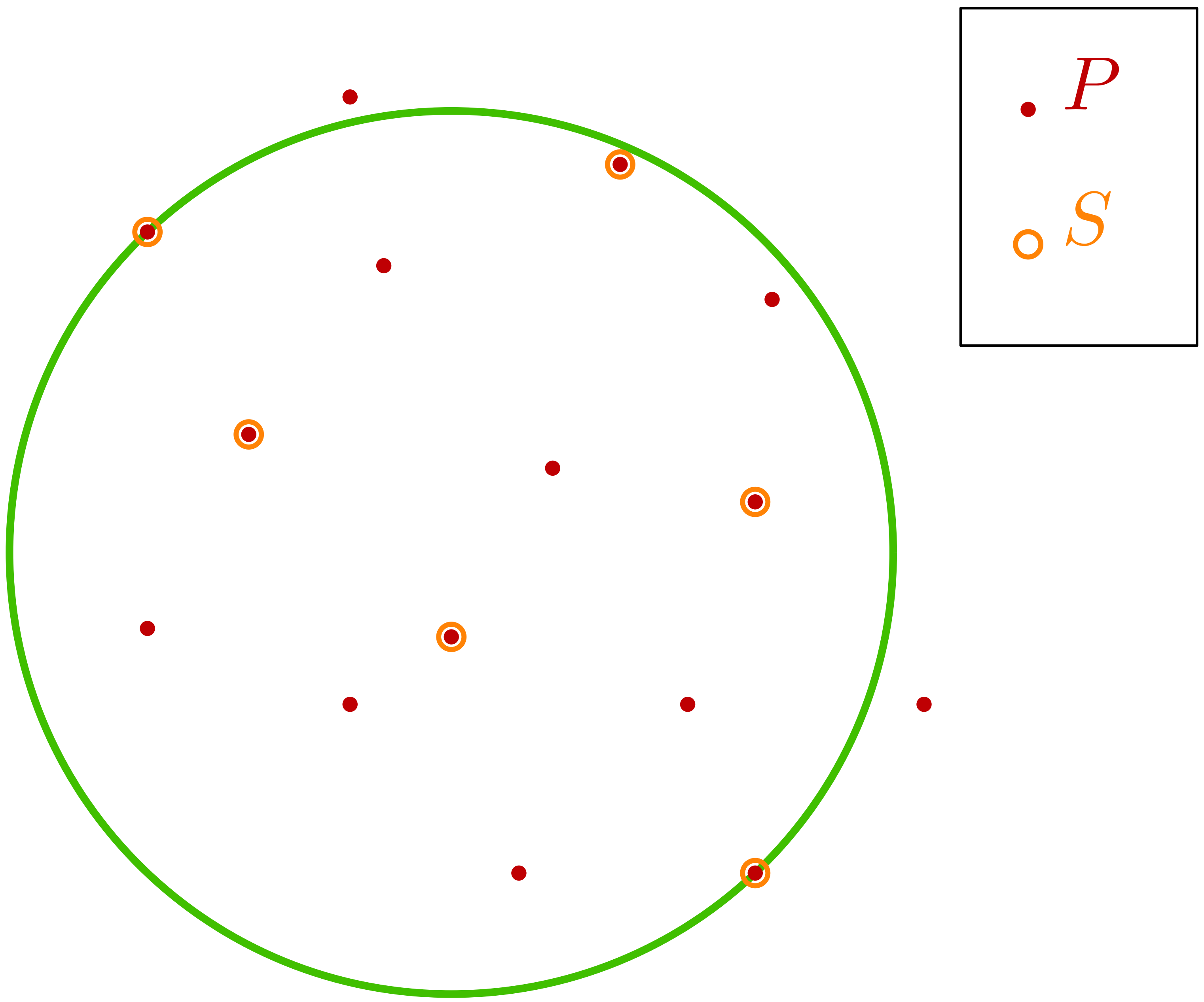

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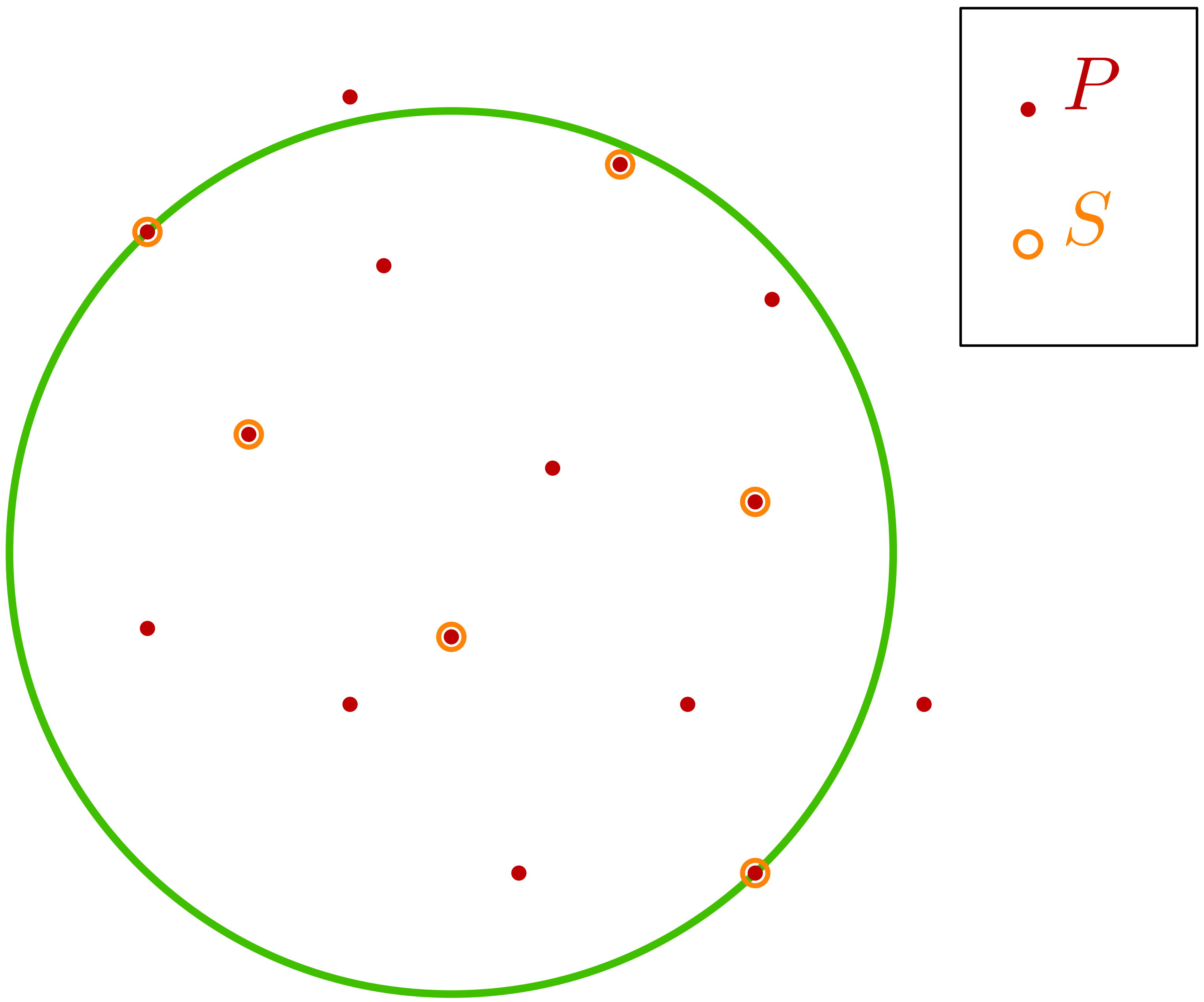

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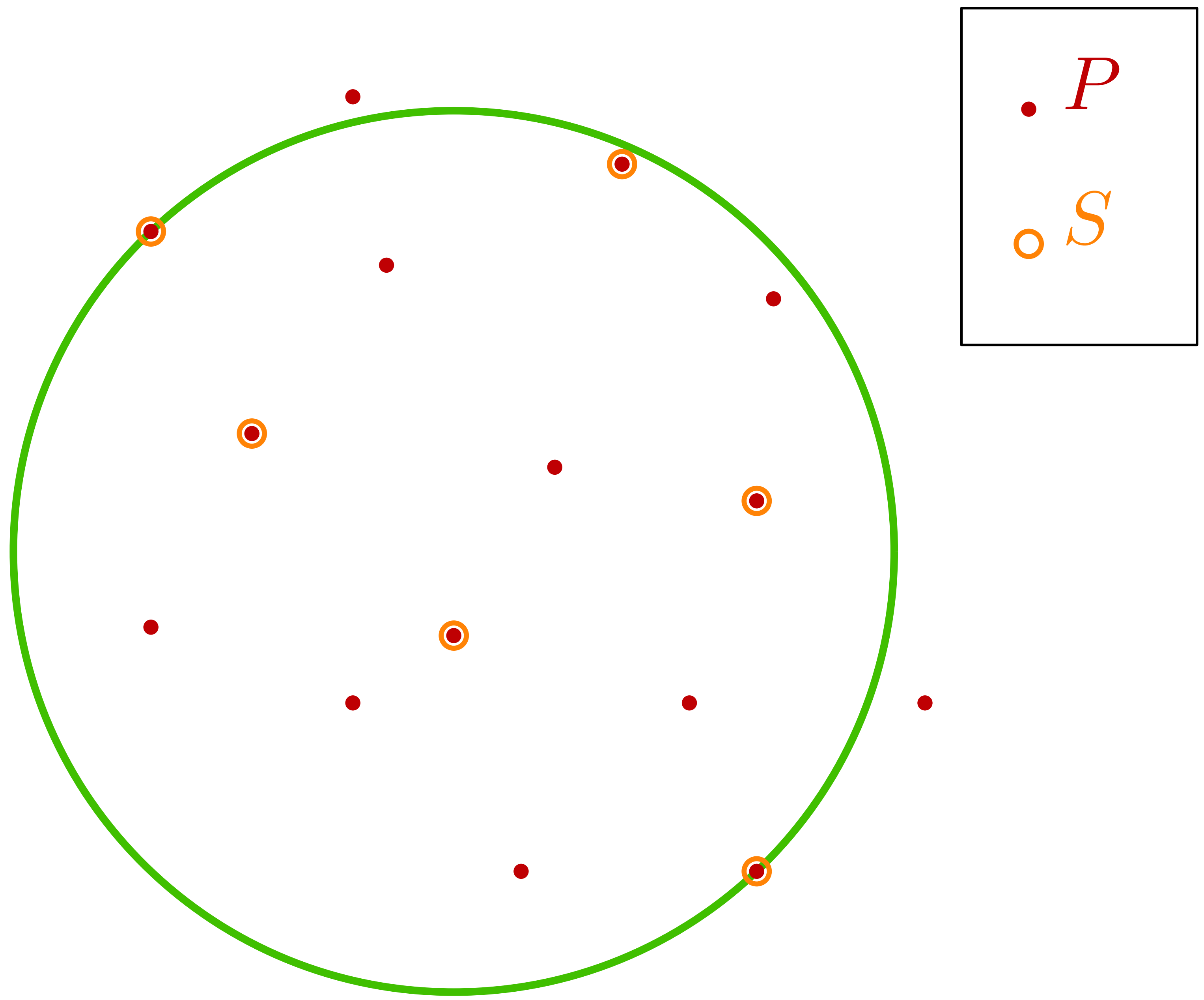

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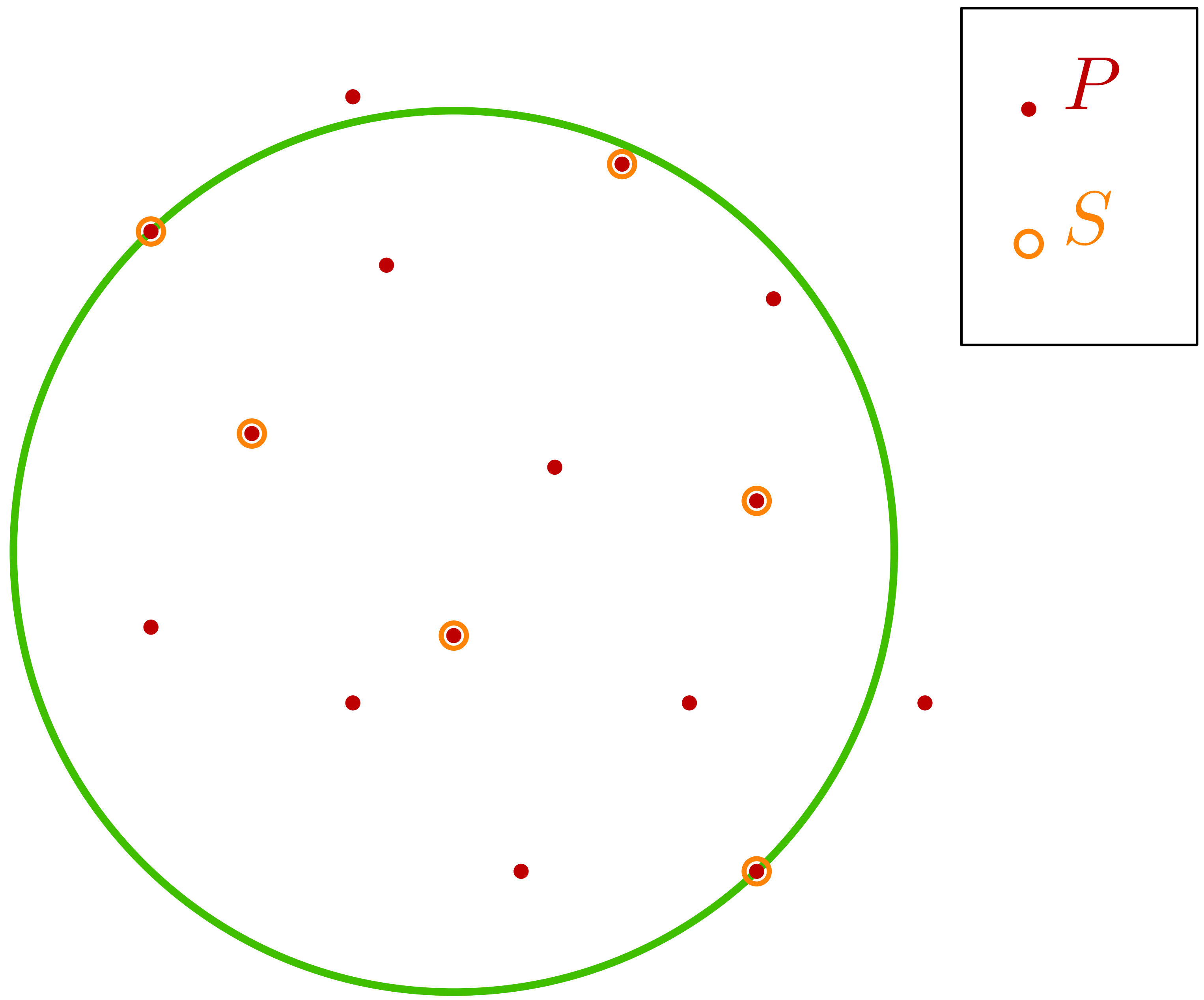

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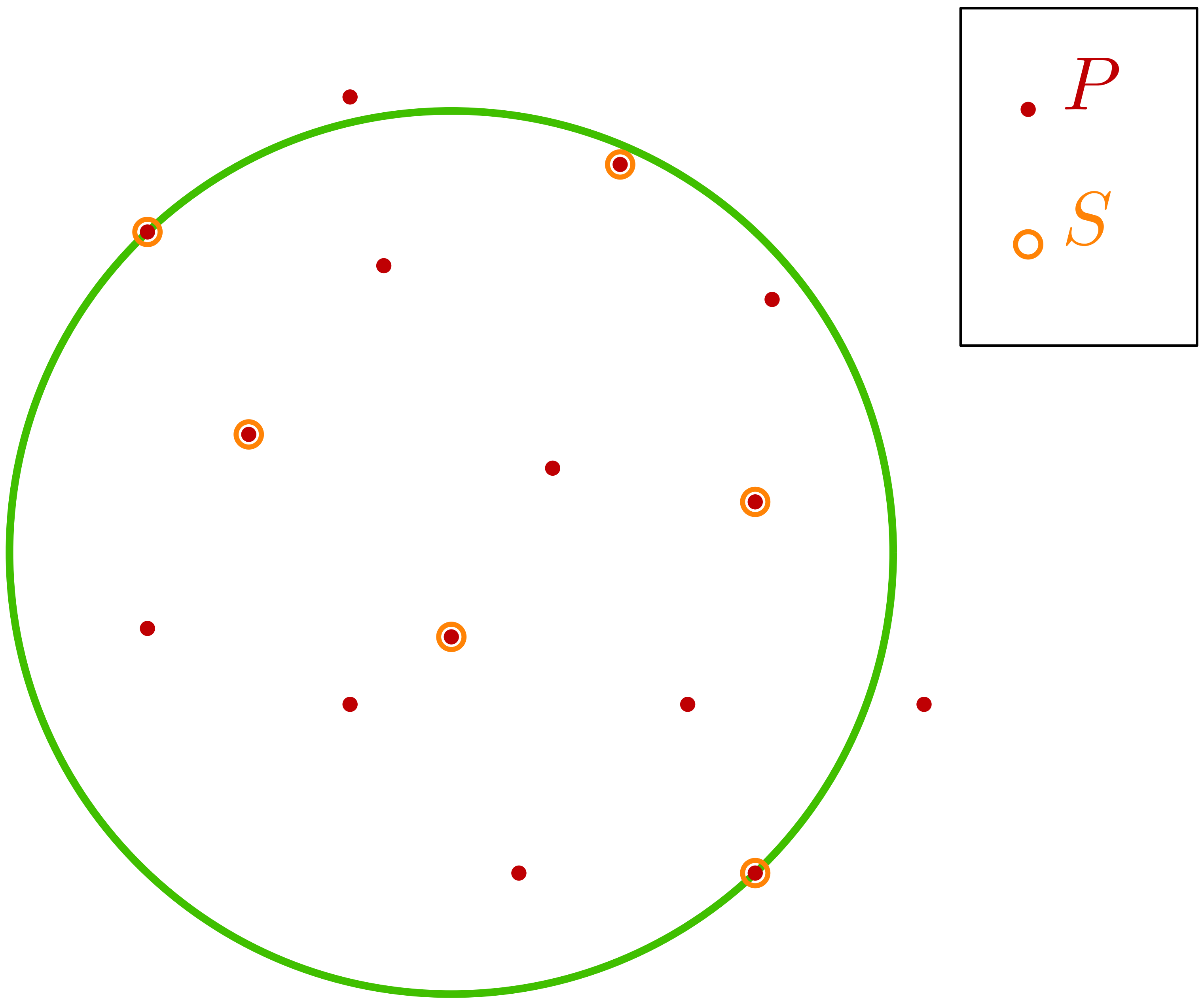

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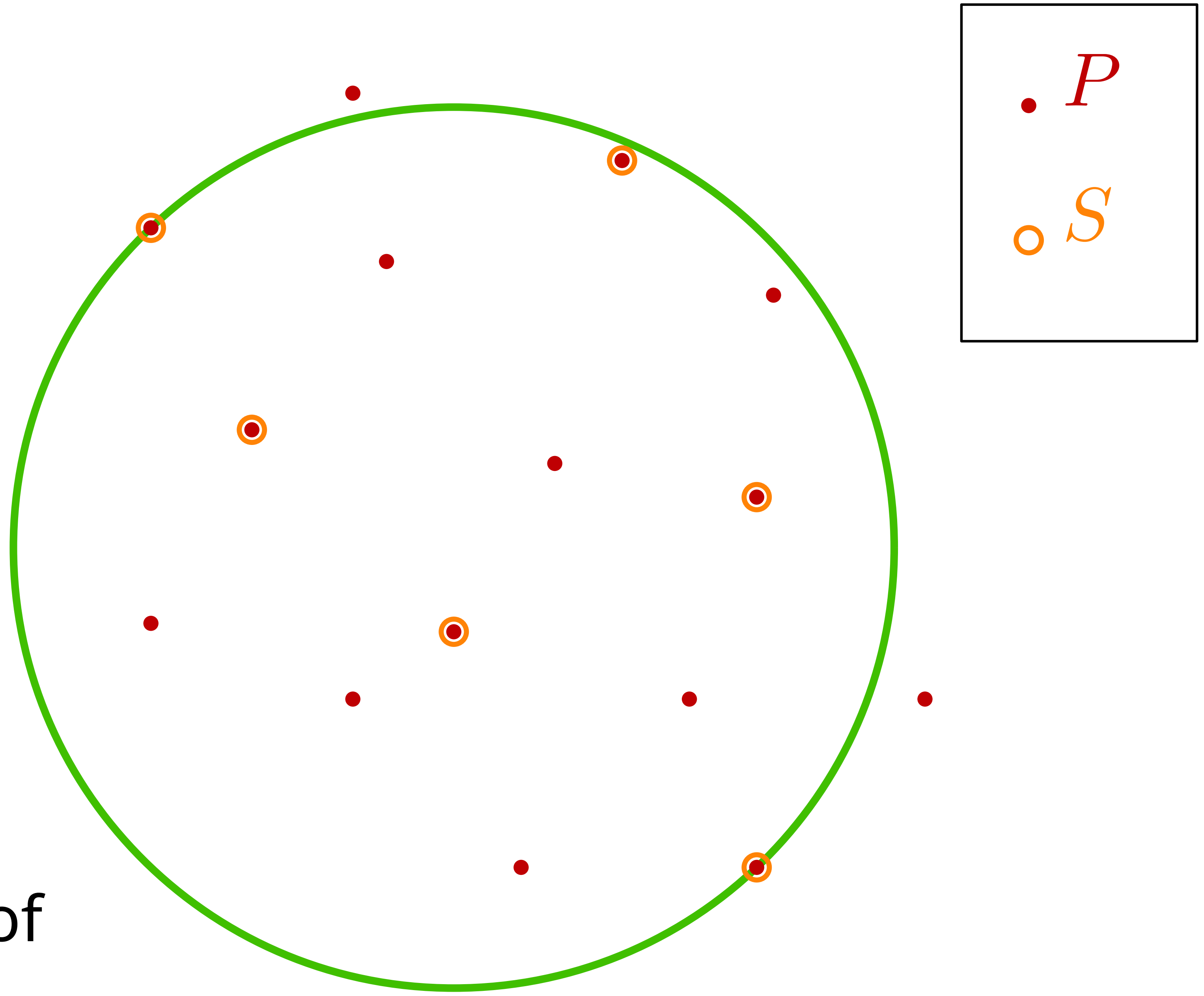

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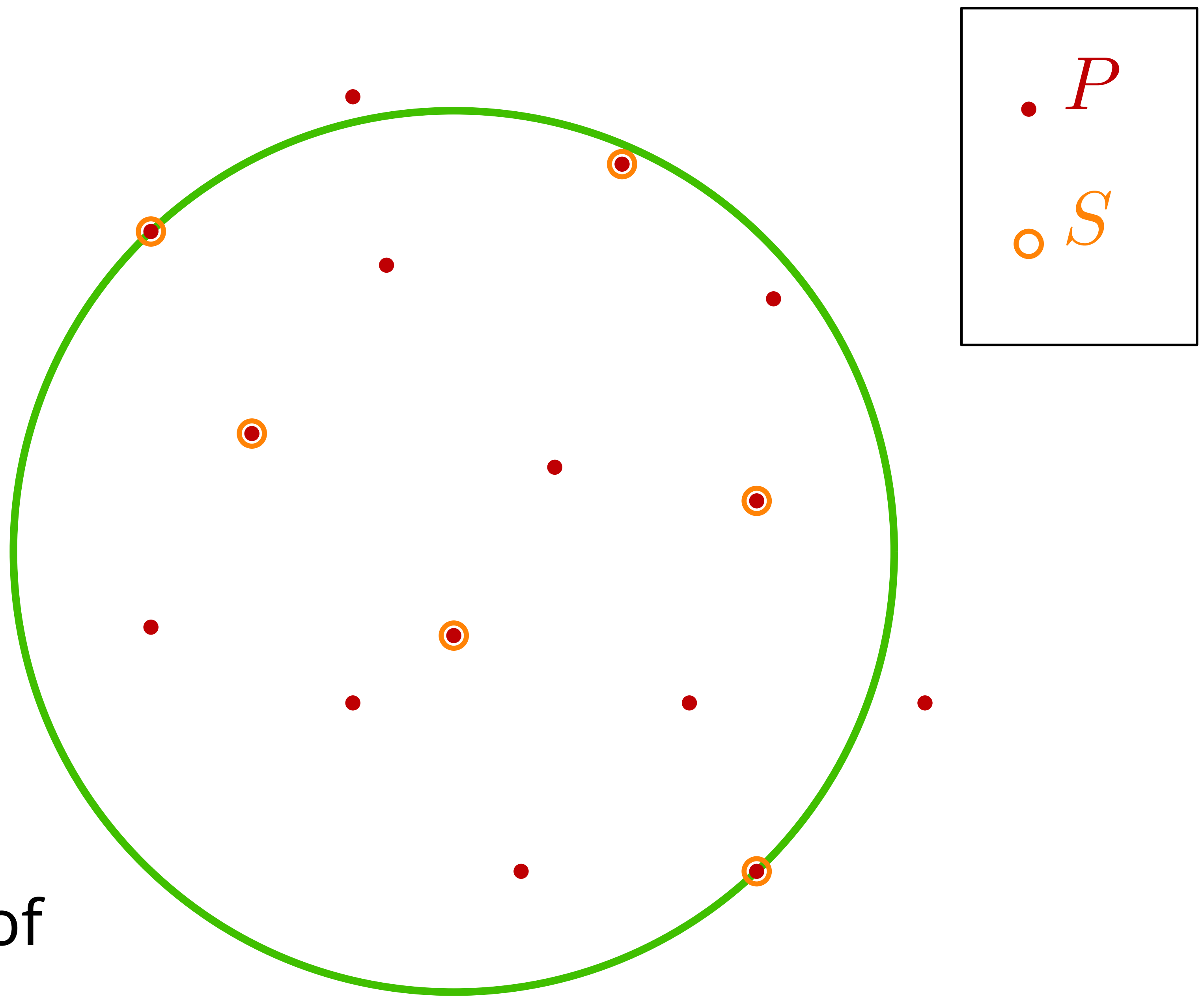

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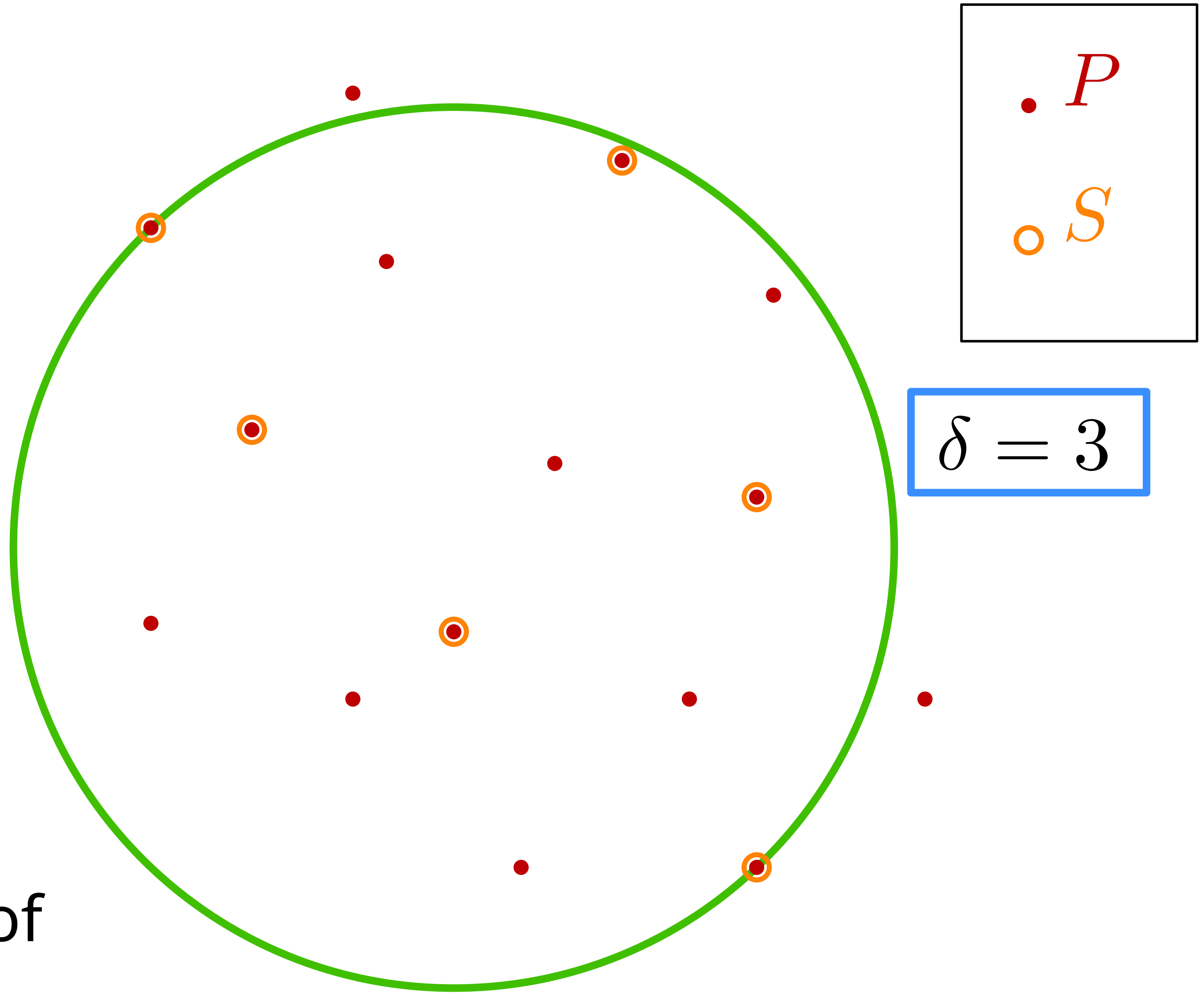

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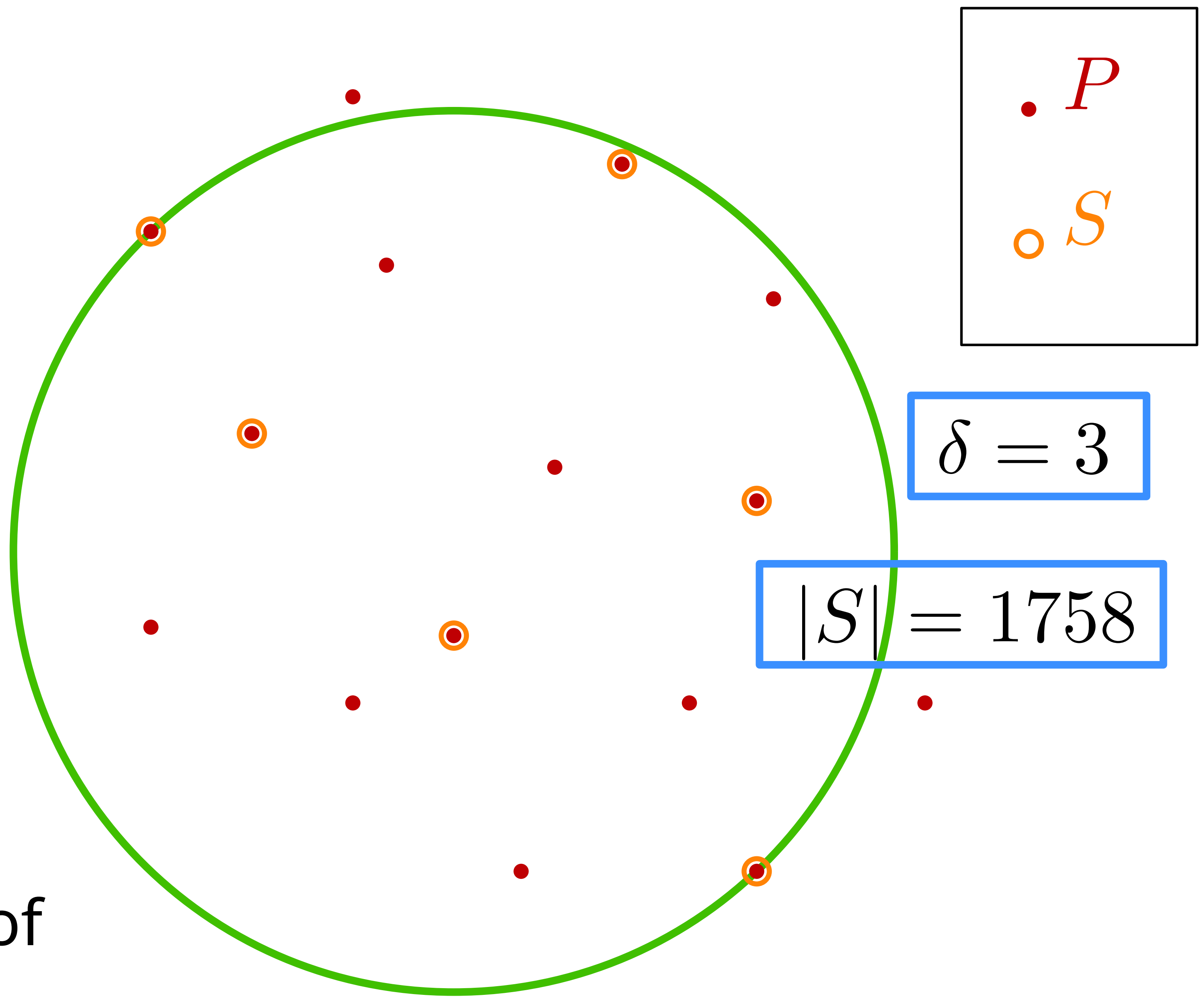

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Question: How large is $\log|\mathcal{R}|$?

Sauer’s Lemma

bounding $|\mathcal{R}|$

Number of small subsets

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Intuition: Take element x: subsets don't contain x or do

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If $(X, \mathcal{R})$ is a range space with VC-dimension d and $|X| = n$, then $|\mathcal{R}| \leq \Phi_d(n)$.

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Base: $d = 0$ and $n = 0$ trivially true

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These are exactly elements in $\mathcal{R}_x$!

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Quiz

Which bound on $O\left(\frac{\log|\mathcal{R}|}{\varepsilon^2}\right)$ does the previous lemma give for $(X, \mathcal{R})$ with $n = |X|$ and VC-dimension δ?

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What does $|\mathcal{R}| = O(n^d)$ imply about the VC-dimension?

Shattering dimension

Shattering Dimension

Given a range space $S = (X, \mathcal{R})$, its shatter function $\pi_S(m)$ is the maximum number of sets that might be created by S when restricted to subsets of size m. Formally,

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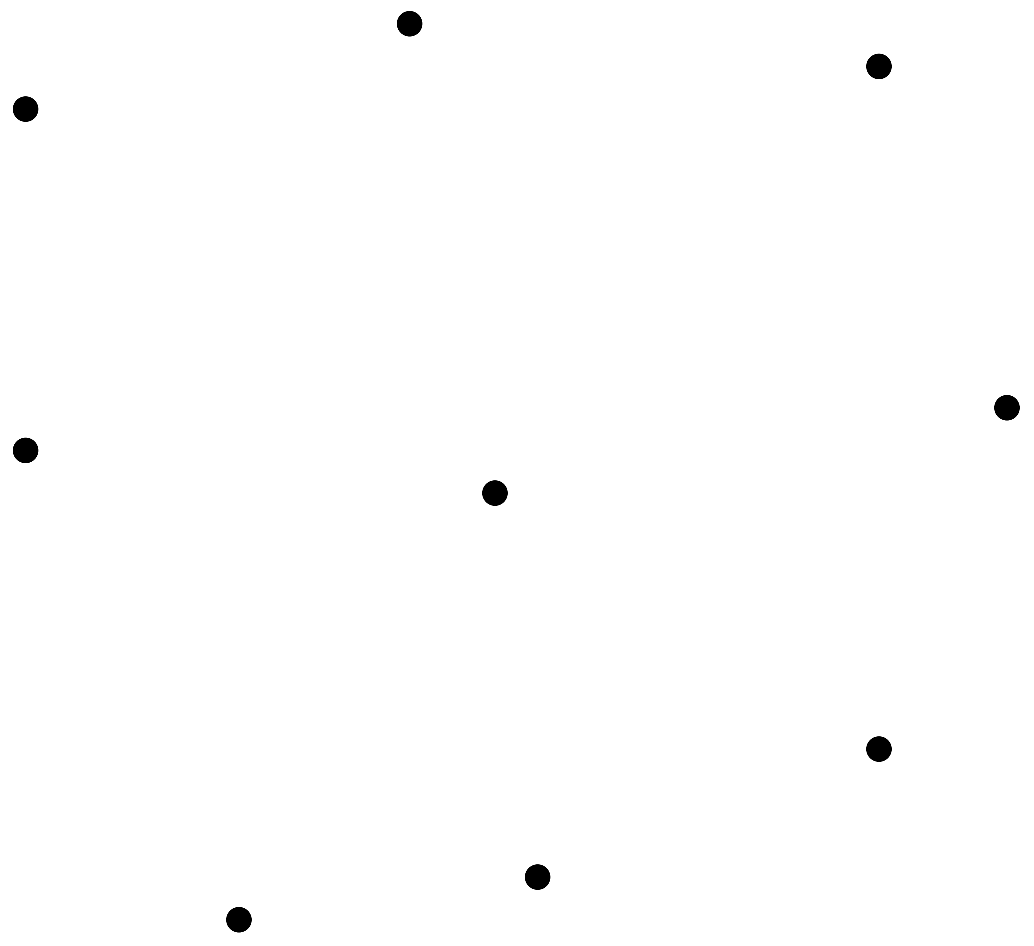
Sauer's lemma: shattering dimension $\leq$ VC-dimension

Examples of Shattering Dimension

range space $(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D} =$ set of disks



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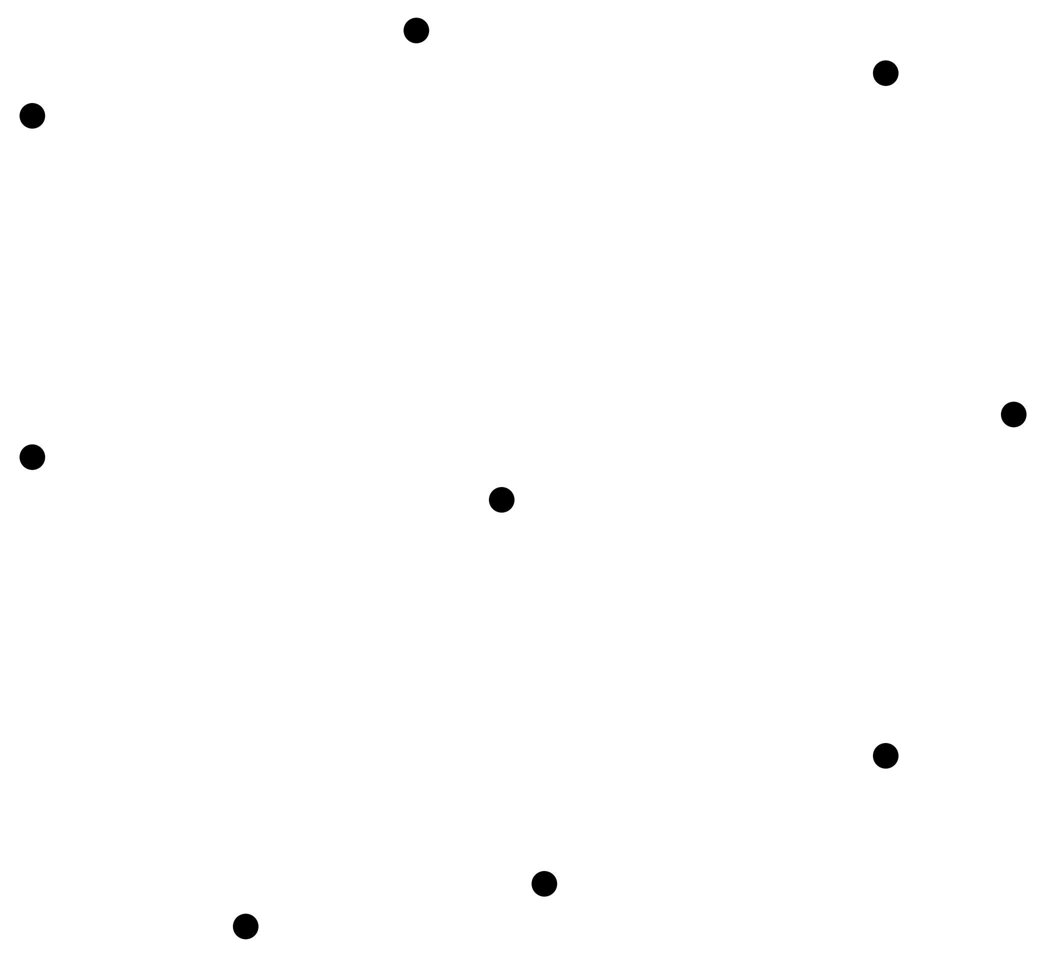

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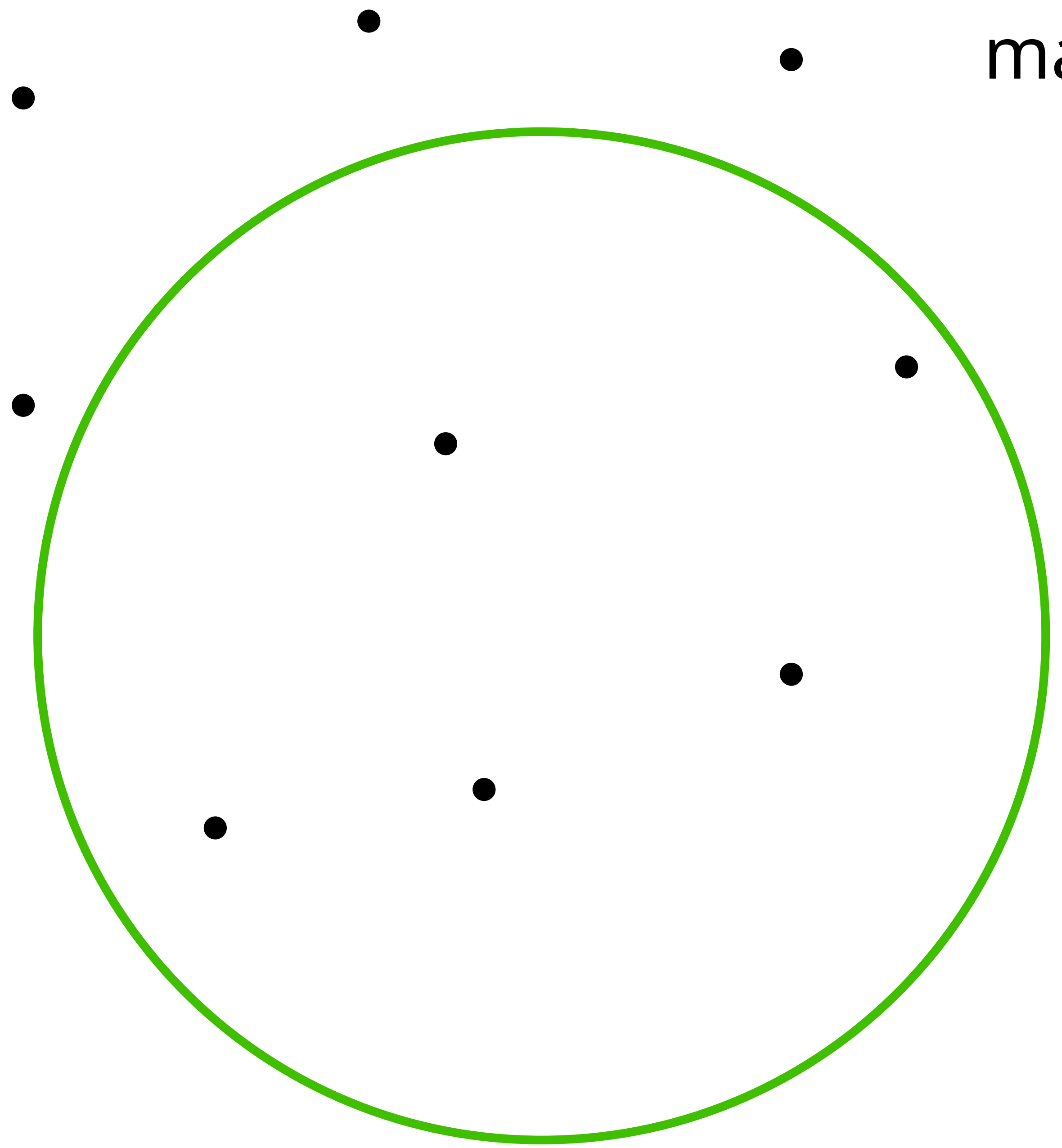

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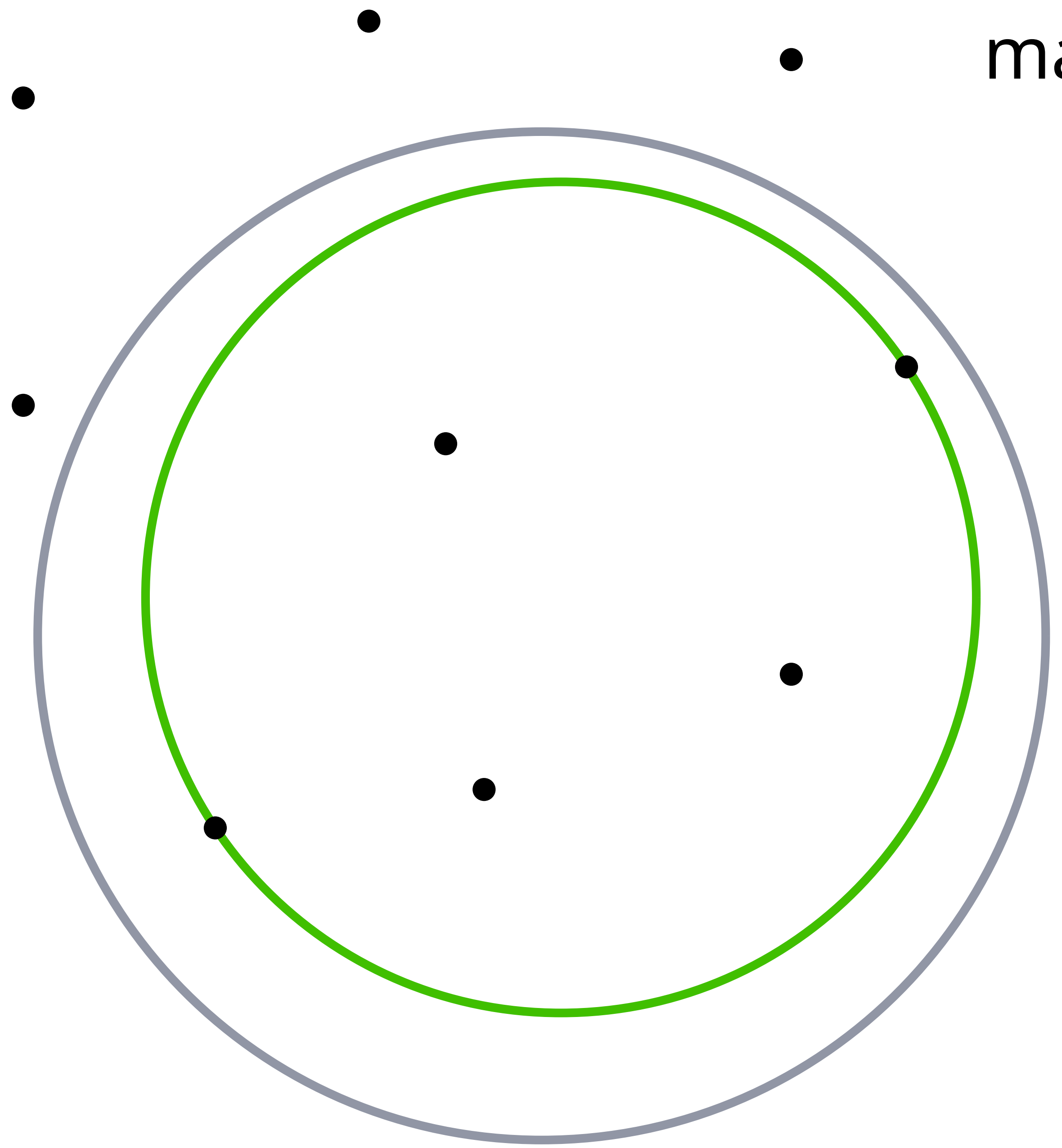

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canonical disk for points in range: smallest enclosing disk

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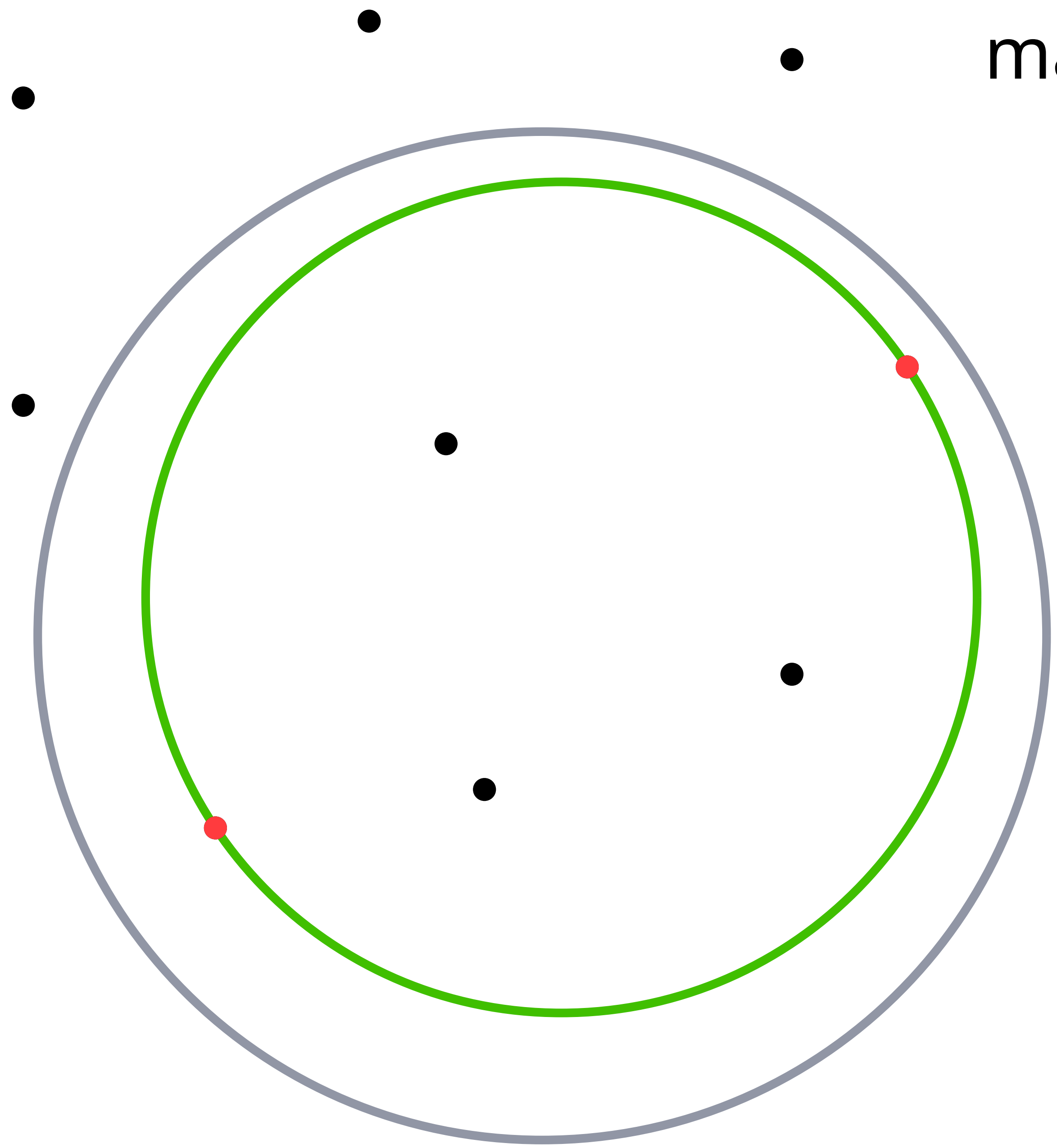

Question: When restricted to n points, how many ranges are there?

consider a disk

canonical disk for points in range: smallest enclosing disk

uniquely determined by ≤ 3 points on boundary

Examples of Shattering Dimension

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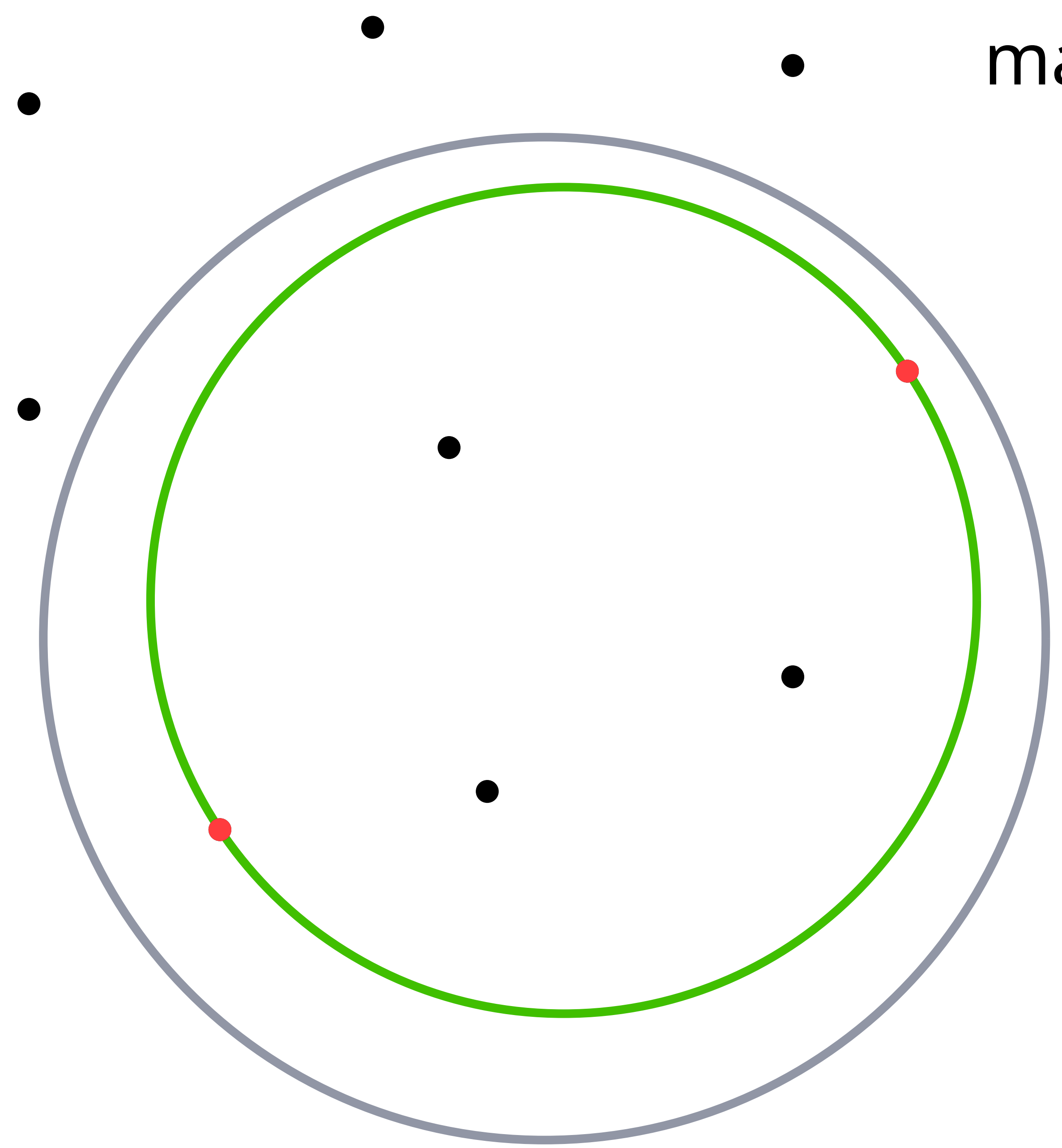

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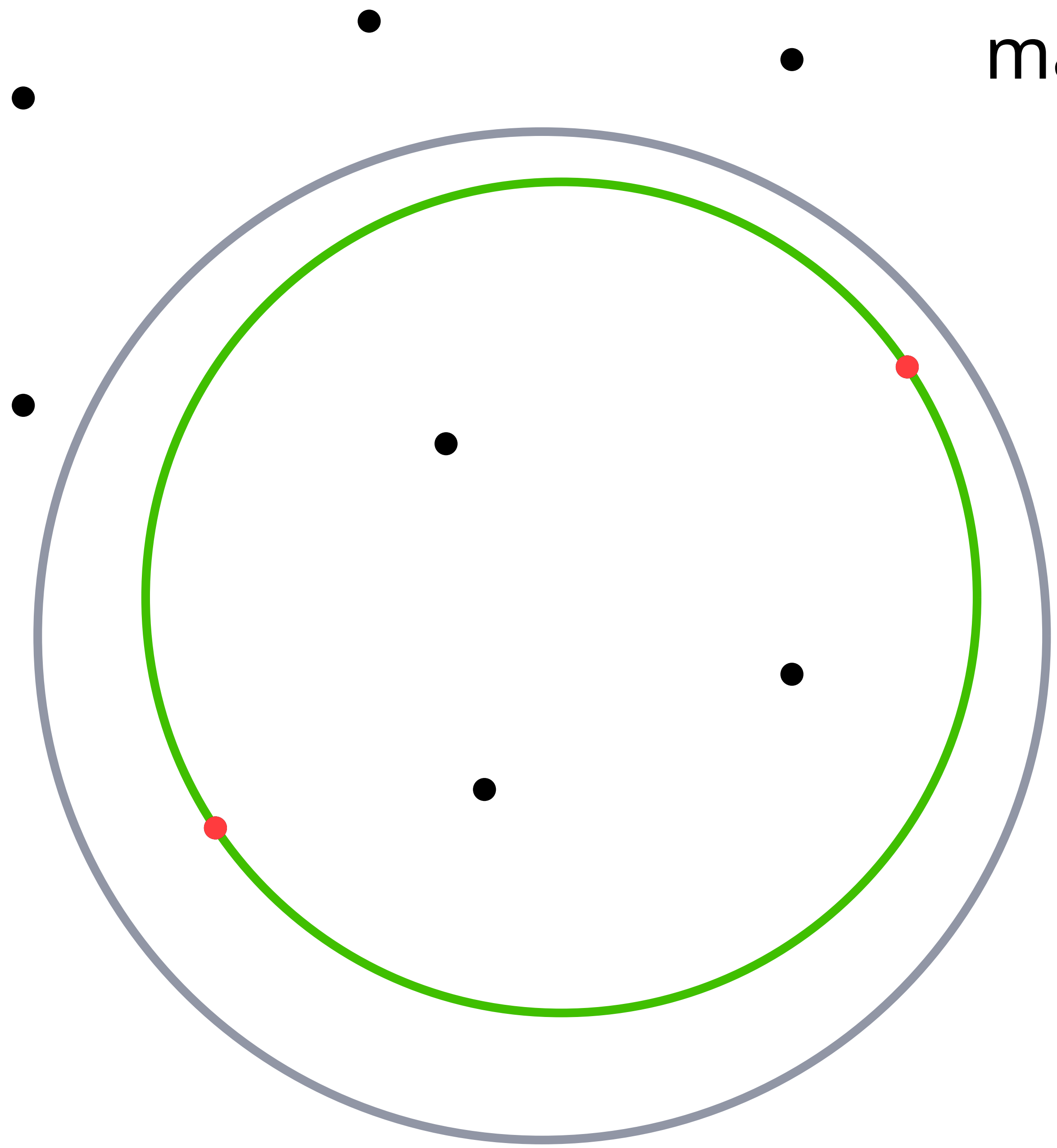

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shattering dim ≤ 3

Shattering dimension of geometric range spaces

shattering dimension $\approx$ how many points determine a range

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range space	shattering dimension
$(\mathbb{R}, \mathcal{I})$, with $\mathcal{I}$ $=$ set of closed intervals ?	
$(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D}$ $=$ set of disks	3
$(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ $=$ set of axis-aligned rectangles	
$(\mathbb{R}^2, \mathcal{GR})$, with $\mathcal{GR}$ $=$ set of arbitrary oriented rectangles	
$(\mathbb{R}^2, \mathcal{C})$, with $\mathcal{C}$ $=$ set of closed convex sets	–

Shattering dimension of geometric range spaces

shattering dimension $\approx$ how many points determine a range

range space	shattering dimension
$(\mathbb{R}, \mathcal{I})$, with $\mathcal{I}$ = set of closed intervals	2
$(\mathbb{R}^2, \mathcal{D})$, with $\mathcal{D}$ = set of disks	3
$(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles ?	
$(\mathbb{R}^2, \mathcal{GR})$, with $\mathcal{GR}$ = set of arbitrary oriented rectangles	
$(\mathbb{R}^2, \mathcal{C})$, with $\mathcal{C}$ = set of closed convex sets	–

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$(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR}$ = set of axis-aligned rectangles	4
$(\mathbb{R}^2, \mathcal{GR})$, with $\mathcal{GR}$ = set of arbitrary oriented rectangles	
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$(\mathbb{R}^2, \mathcal{AR})$, with $\mathcal{AR} =$ set of axis-aligned rectangles	4
$(\mathbb{R}^2, \mathcal{GR})$, with $\mathcal{GR} =$ set of arbitrary oriented rectangles	5
$(\mathbb{R}^2, \mathcal{C})$, with $\mathcal{C} =$ set of closed convex sets	–

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Can be easier to compute than VC-dimension

Shattering dimension vs VC-dimension

VC-dimension δ

shattering dimension d

Shattering dimension vs VC-dimension

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Sauer's lemma: $d \leq \delta$

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Summary

range space $(X, \mathcal{R})$

VC-dimension δ

examples of geometric range spaces

ε-sample of size $O\left(\frac{\delta+\log(\varphi^{-1})}{\varepsilon^2}\right)$

ε-net of size $O\left(\frac{\delta \log \varepsilon^{-1}+\log(\varphi^{-1})}{\varepsilon}\right)$

applications for geometric approximation

shattering dimension d

$d \leq \delta \leq d \log d$